

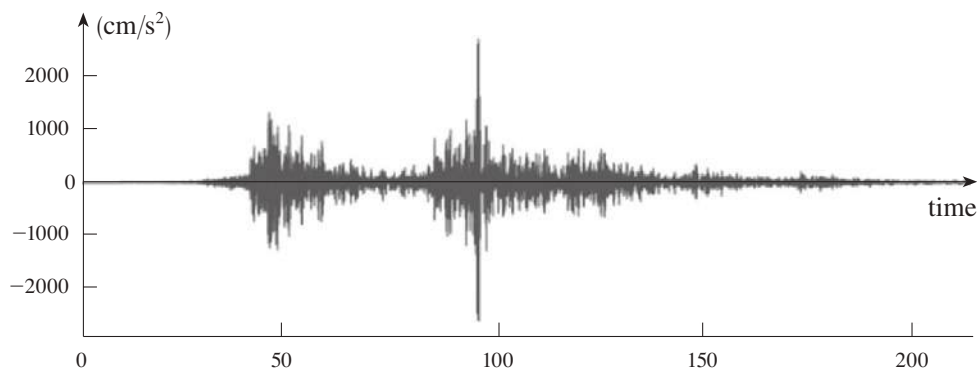
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Functions and Limits

Often a graph is the best way to represent a function because it conveys so much information at a glance. Shown is a graph of the vertical ground acceleration created by the 2011 earthquake near Tohoku, Japan. The earthquake had a magnitude of 9.0 on the Richter scale and was so powerful that it moved northern Japan 8 feet closer to North America.



Pictura Collectus/Alamy



Seismological Society of America

THE FUNDAMENTAL OBJECTS THAT WE deal with in calculus are functions. We stress that a function can be represented in different ways: by an equation, in a table, by a graph, or in words. We look at the main types of functions that occur in calculus and describe the process of using these functions as mathematical models of real-world phenomena.

In *A Preview of Calculus* (page 1) we saw how the idea of a limit underlies the various branches of calculus. It is therefore appropriate to begin our study of calculus by investigating limits of functions and their properties.

1.1 Four Ways to Represent a Function

Functions arise whenever one quantity depends on another. Consider the following four situations.

- A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A , and we say that A is a *function* of r .
- B. The human population of the world P depends on the time t . The table gives estimates of the world population $P(t)$ at time t , for certain years. For instance,

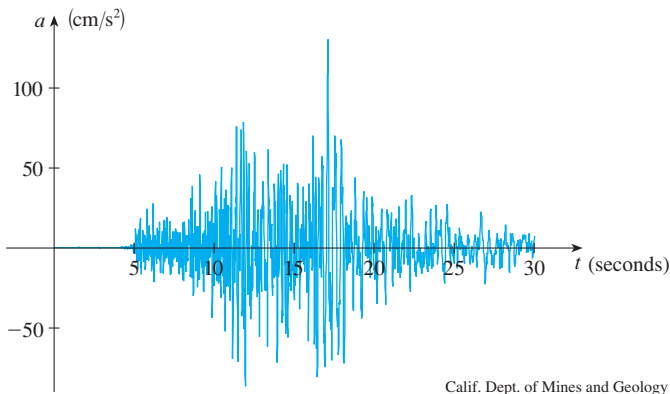
$$P(1950) \approx 2,560,000,000$$

But for each value of the time t there is a corresponding value of P , and we say that P is a function of t .

- C. The cost C of mailing an envelope depends on its weight w . Although there is no simple formula that connects w and C , the post office has a rule for determining C when w is known.
- D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t . Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t , the graph provides a corresponding value of a .

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

FIGURE 1
Vertical ground acceleration
during the Northridge earthquake



Each of these examples describes a rule whereby, given a number (r , t , w , or t), another number (A , P , C , or a) is assigned. In each case we say that the second number is a function of the first number.

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E .

We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain** of the function. The number $f(x)$ is the **value of f at x** and is read “ f of x .” The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain. A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. A symbol that represents a number in the *range* of f is called a **dependent variable**. In Example A, for instance, r is the independent variable and A is the dependent variable.



FIGURE 2

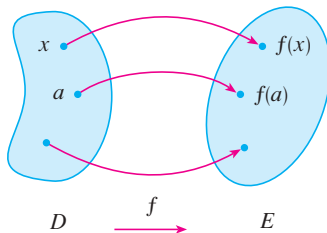
Machine diagram for a function f 

FIGURE 3

Arrow diagram for f

It's helpful to think of a function as a **machine** (see Figure 2). If x is in the domain of the function f , then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

The preprogrammed functions in a calculator are good examples of a function as a machine. For example, the square root key on your calculator computes such a function. You press the key labeled $\sqrt{}$ (or $\sqrt{x}$) and enter the input x . If $x < 0$, then x is not in the domain of this function; that is, x is not an acceptable input, and the calculator will indicate an error. If $x \geq 0$, then an *approximation* to $\sqrt{x}$ will appear in the display. Thus the $\sqrt{x}$ key on your calculator is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$.

Another way to picture a function is by an **arrow diagram** as in Figure 3. Each arrow connects an element of D to an element of E . The arrow indicates that $f(x)$ is associated with x , $f(a)$ is associated with a , and so on.

The most common method for visualizing a function is its graph. If f is a function with domain D , then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f .

The graph of a function f gives us a useful picture of the behavior or “life history” of a function. Since the y -coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 4). The graph of f also allows us to picture the domain of f on the x -axis and its range on the y -axis as in Figure 5.

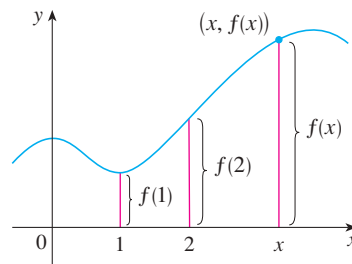


FIGURE 4

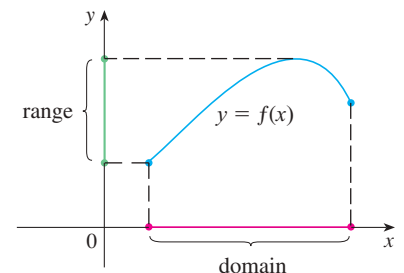


FIGURE 5

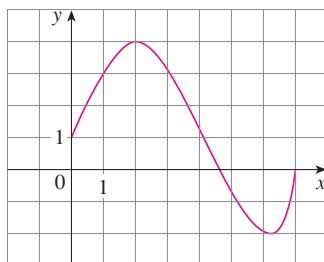


FIGURE 6

The notation for intervals is given in Appendix A.

EXAMPLE 1 The graph of a function f is shown in Figure 6.

- Find the values of $f(1)$ and $f(5)$.
- What are the domain and range of f ?

SOLUTION

- We see from Figure 6 that the point $(1, 3)$ lies on the graph of f , so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x -axis.)

When $x = 5$, the graph lies about 0.7 units below the x -axis, so we estimate that $f(5) \approx -0.7$.

- We see that $f(x)$ is defined when $0 \leq x \leq 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4 , so the range of f is

$$\{y \mid -2 \leq y \leq 4\} = [-2, 4]$$

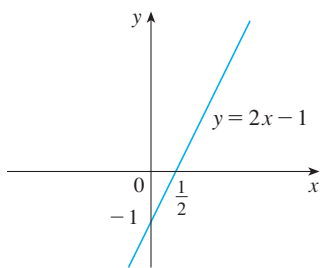


FIGURE 7

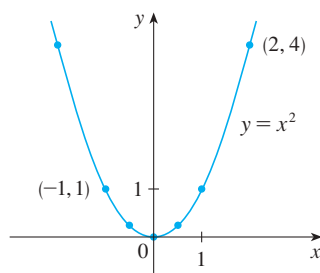


FIGURE 8

EXAMPLE 2 Sketch the graph and find the domain and range of each function.

(a) $f(x) = 2x - 1$

(b) $g(x) = x^2$

SOLUTION

(a) The equation of the graph is $y = 2x - 1$, and we recognize this as being the equation of a line with slope 2 and y-intercept -1 . (Recall the slope-intercept form of the equation of a line: $y = mx + b$. See Appendix B.) This enables us to sketch a portion of the graph of f in Figure 7. The expression $2x - 1$ is defined for all real numbers, so the domain of f is the set of all real numbers, which we denote by $\mathbb{R}$. The graph shows that the range is also $\mathbb{R}$.

(b) Since $g(2) = 2^2 = 4$ and $g(-1) = (-1)^2 = 1$, we could plot the points $(2, 4)$ and $(-1, 1)$, together with a few other points on the graph, and join them to produce the graph (Figure 8). The equation of the graph is $y = x^2$, which represents a parabola (see Appendix C). The domain of g is $\mathbb{R}$. The range of g consists of all values of $g(x)$, that is, all numbers of the form x^2 . But $x^2 \geq 0$ for all numbers x and any positive number y is a square. So the range of g is $\{y \mid y \geq 0\} = [0, \infty)$. This can also be seen from Figure 8. ■

EXAMPLE 3 If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\frac{f(a+h) - f(a)}{h}$.

SOLUTION We first evaluate $f(a+h)$ by replacing x by $a+h$ in the expression for $f(x)$:

$$\begin{aligned} f(a+h) &= 2(a+h)^2 - 5(a+h) + 1 \\ &= 2(a^2 + 2ah + h^2) - 5(a+h) + 1 \\ &= 2a^2 + 4ah + 2h^2 - 5a - 5h + 1 \end{aligned}$$

Then we substitute into the given expression and simplify:

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 - 5a - 5h + 1) - (2a^2 - 5a + 1)}{h} \\ &= \frac{2a^2 + 4ah + 2h^2 - 5a - 5h + 1 - 2a^2 + 5a - 1}{h} \\ &= \frac{4ah + 2h^2 - 5h}{h} = 4a + 2h - 5 \end{aligned}$$

The expression

$$\frac{f(a+h) - f(a)}{h}$$

in Example 3 is called a **difference quotient** and occurs frequently in calculus. As we will see in Chapter 2, it represents the average rate of change of $f(x)$ between $x = a$ and $x = a + h$.

■ Representations of Functions

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

If a single function can be represented in all four ways, it's often useful to go from one representation to another to gain additional insight into the function. (In Example 2, for instance, we started with algebraic formulas and then obtained the graphs.) But certain functions are described more naturally by one method than by another. With this in mind, let's reexamine the four situations that we considered at the beginning of this section.

- A. The most useful representation of the area of a circle as a function of its radius is probably the algebraic formula $A(r) = \pi r^2$, though it is possible to compile a table of values or to sketch a graph (half a parabola). Because a circle has to have a positive radius, the domain is $\{r \mid r > 0\} = (0, \infty)$, and the range is also $(0, \infty)$.
- B. We are given a description of the function in words: $P(t)$ is the human population of the world at time t . Let's measure t so that $t = 0$ corresponds to the year 1900. The table of values of world population provides a convenient representation of this function. If we plot these values, we get the graph (called a *scatter plot*) in Figure 9. It too is a useful representation; the graph allows us to absorb all the data at once. What about a formula? Of course, it's impossible to devise an explicit formula that gives the exact human population $P(t)$ at any time t . But it is possible to find an expression for a function that *approximates* $P(t)$. In fact, using methods explained in Section 1.2, we obtain the approximation

$$P(t) \approx f(t) = (1.43653 \times 10^9) \cdot (1.01395)^t$$

Figure 10 shows that it is a reasonably good “fit.” The function f is called a *mathematical model* for population growth. In other words, it is a function with an explicit formula that approximates the behavior of our given function. We will see, however, that the ideas of calculus can be applied to a table of values; an explicit formula is not necessary.

t (years since 1900)	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

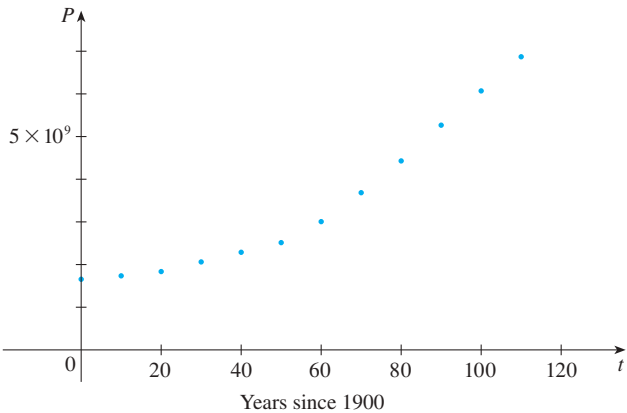


FIGURE 9

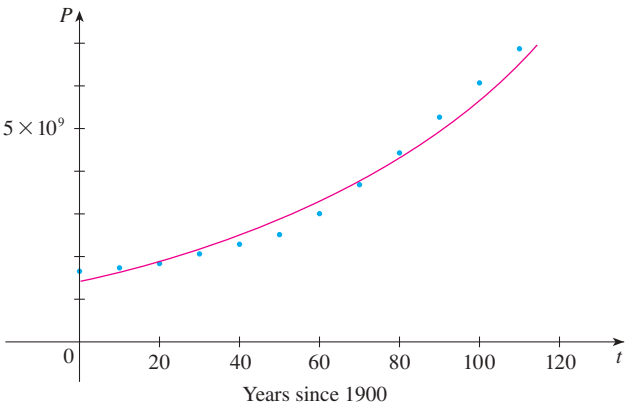


FIGURE 10

A function defined by a table of values is called a *tabular* function.

w (ounces)	$C(w)$ (dollars)
$0 < w \leq 1$	0.98
$1 < w \leq 2$	1.19
$2 < w \leq 3$	1.40
$3 < w \leq 4$	1.61
$4 < w \leq 5$	1.82
$\vdots$	$\vdots$
$\vdots$	$\vdots$

- The function P is typical of the functions that arise whenever we attempt to apply calculus to the real world. We start with a verbal description of a function. Then we may be able to construct a table of values of the function, perhaps from instrument readings in a scientific experiment. Even though we don't have complete knowledge of the values of the function, we will see throughout the book that it is still possible to perform the operations of calculus on such a function.
- C. Again the function is described in words: Let $C(w)$ be the cost of mailing a large envelope with weight w . The rule that the US Postal Service used as of 2015 is as follows: The cost is 98 cents for up to 1 oz, plus 21 cents for each additional ounce (or less) up to 13 oz. The table of values shown in the margin is the most convenient representation for this function, though it is possible to sketch a graph (see Example 10).
- D. The graph shown in Figure 1 is the most natural representation of the vertical acceleration function $a(t)$. It's true that a table of values could be compiled, and it is even possible to devise an approximate formula. But everything a geologist needs to

know—amplitudes and patterns—can be seen easily from the graph. (The same is true for the patterns seen in electrocardiograms of heart patients and polygraphs for lie-detection.)

In the next example we sketch the graph of a function that is defined verbally.

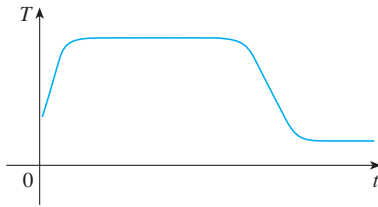


FIGURE 11

EXAMPLE 4 When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

SOLUTION The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the hot-water tank starts flowing from the faucet, T increases quickly. In the next phase, T is constant at the temperature of the heated water in the tank. When the tank is drained, T decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t in Figure 11. ■

In the following example we start with a verbal description of a function in a physical situation and obtain an explicit algebraic formula. The ability to do this is a useful skill in solving calculus problems that ask for the maximum or minimum values of quantities.

EXAMPLE 5 A rectangular storage container with an open top has a volume of 10 m^3 . The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

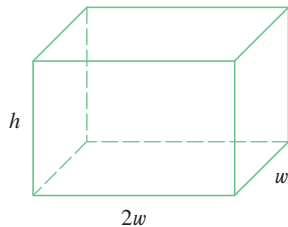


FIGURE 12

SOLUTION We draw a diagram as in Figure 12 and introduce notation by letting w and $2w$ be the width and length of the base, respectively, and h be the height.

The area of the base is $(2w)w = 2w^2$, so the cost, in dollars, of the material for the base is $10(2w^2)$. Two of the sides have area wh and the other two have area $2wh$, so the cost of the material for the sides is $6[2(wh) + 2(2wh)]$. The total cost is therefore

$$C = 10(2w^2) + 6[2(wh) + 2(2wh)] = 20w^2 + 36wh$$

To express C as a function of w alone, we need to eliminate h and we do so by using the fact that the volume is 10 m^3 . Thus

$$w(2w)h = 10$$

which gives

$$h = \frac{10}{2w^2} = \frac{5}{w^2}$$

Substituting this into the expression for C , we have

$$C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}$$

Therefore the equation

$$C(w) = 20w^2 + \frac{180}{w} \quad w > 0$$

expresses C as a function of w . ■

EXAMPLE 6 Find the domain of each function.

(a) $f(x) = \sqrt{x+2}$

(b) $g(x) = \frac{1}{x^2 - x}$

PS In setting up applied functions as in Example 5, it may be useful to review the principles of problem solving as discussed on page 98, particularly *Step 1: Understand the Problem*.

Domain Convention

If a function is given by a formula and the domain is not stated explicitly, the convention is that the domain is the set of all numbers for which the formula makes sense and defines a real number.

SOLUTION

(a) Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x + 2 \geq 0$. This is equivalent to $x \geq -2$, so the domain is the interval $[-2, \infty)$.

(b) Since

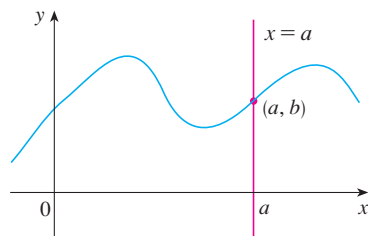
$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$

and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. Thus the domain of g is

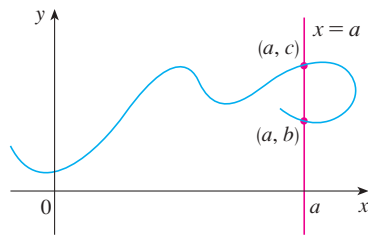
$$\{x \mid x \neq 0, x \neq 1\}$$

which could also be written in interval notation as

$$(-\infty, 0) \cup (0, 1) \cup (1, \infty)$$



(a) This curve represents a function.



(b) This curve doesn't represent a function.

FIGURE 13

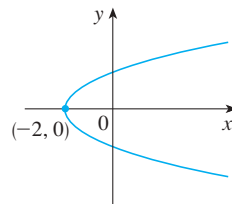
The graph of a function is a curve in the xy -plane. But the question arises: which curves in the xy -plane are graphs of functions? This is answered by the following test.

The Vertical Line Test A curve in the xy -plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

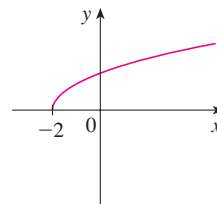
The reason for the truth of the Vertical Line Test can be seen in Figure 13. If each vertical line $x = a$ intersects a curve only once, at (a, b) , then exactly one function value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c) , then the curve can't represent a function because a function can't assign two different values to a .

For example, the parabola $x = y^2 - 2$ shown in Figure 14(a) is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice. The parabola, however, does contain the graphs of *two* functions of x . Notice that the equation $x = y^2 - 2$ implies $y^2 = x + 2$, so $y = \pm\sqrt{x + 2}$. Thus the upper and lower halves of the parabola are the graphs of the functions $f(x) = \sqrt{x + 2}$ [from Example 6(a)] and $g(x) = -\sqrt{x + 2}$. [See Figures 14(b) and (c).]

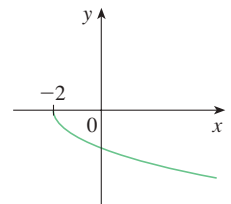
We observe that if we reverse the roles of x and y , then the equation $x = h(y) = y^2 - 2$ *does* define x as a function of y (with y as the independent variable and x as the dependent variable) and the parabola now appears as the graph of the function h .



(a) $x = y^2 - 2$



(b) $y = \sqrt{x + 2}$



(c) $y = -\sqrt{x + 2}$

FIGURE 14

■ **Piecewise Defined Functions**

The functions in the following four examples are defined by different formulas in different parts of their domains. Such functions are called **piecewise defined functions**.

EXAMPLE 7 A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

SOLUTION Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x . If it happens that $x \leq -1$, then the value of $f(x)$ is $1 - x$. On the other hand, if $x > -1$, then the value of $f(x)$ is x^2 .

Since $-2 \leq -1$, we have $f(-2) = 1 - (-2) = 3$.

Since $-1 \leq -1$, we have $f(-1) = 1 - (-1) = 2$.

Since $0 > -1$, we have $f(0) = 0^2 = 0$.

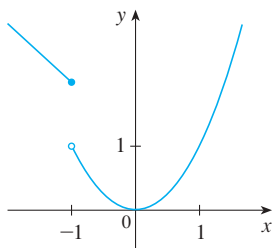


FIGURE 15

How do we draw the graph of f ? We observe that if $x \leq -1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of the vertical line $x = -1$ must coincide with the line $y = 1 - x$, which has slope -1 and y -intercept 1 . If $x > -1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = -1$ must coincide with the graph of $y = x^2$, which is a parabola. This enables us to sketch the graph in Figure 15. The solid dot indicates that the point $(-1, 2)$ is included on the graph; the open dot indicates that the point $(-1, 1)$ is excluded from the graph. ■

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number a , denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0 , so we have

$$|a| \geq 0 \quad \text{for every number } a$$

For example,

$$|3| = 3 \quad |-3| = 3 \quad |0| = 0 \quad |\sqrt{2} - 1| = \sqrt{2} - 1 \quad |3 - \pi| = \pi - 3$$

In general, we have

$$\begin{aligned} |a| &= a & \text{if } a \geq 0 \\ |a| &= -a & \text{if } a < 0 \end{aligned}$$

(Remember that if a is negative, then $-a$ is positive.)

EXAMPLE 8 Sketch the graph of the absolute value function $f(x) = |x|$.

SOLUTION From the preceding discussion we know that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using the same method as in Example 7, we see that the graph of f coincides with the line $y = x$ to the right of the y -axis and coincides with the line $y = -x$ to the left of the y -axis (see Figure 16). ■

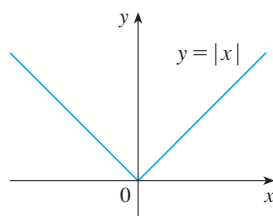


FIGURE 16

For a more extensive review of absolute values, see Appendix A.

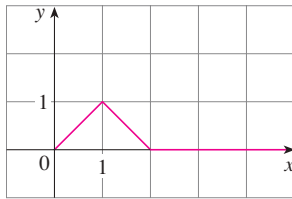


FIGURE 17

Point-slope form of the equation of a line:

$$y - y_1 = m(x - x_1)$$

See Appendix B.

EXAMPLE 9 Find a formula for the function f graphed in Figure 17.

SOLUTION The line through $(0, 0)$ and $(1, 1)$ has slope $m = 1$ and y -intercept $b = 0$, so its equation is $y = x$. Thus, for the part of the graph of f that joins $(0, 0)$ to $(1, 1)$, we have

$$f(x) = x \quad \text{if } 0 \leq x \leq 1$$

The line through $(1, 1)$ and $(2, 0)$ has slope $m = -1$, so its point-slope form is

$$y - 0 = (-1)(x - 2) \quad \text{or} \quad y = 2 - x$$

So we have

$$f(x) = 2 - x \quad \text{if } 1 < x \leq 2$$

We also see that the graph of f coincides with the x -axis for $x > 2$. Putting this information together, we have the following three-piece formula for f :

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \\ 0 & \text{if } x > 2 \end{cases}$$

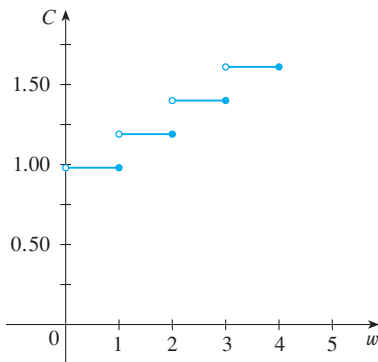


FIGURE 18

EXAMPLE 10 In Example C at the beginning of this section we considered the cost $C(w)$ of mailing a large envelope with weight w . In effect, this is a piecewise defined function because, from the table of values on page 13, we have

$$C(w) = \begin{cases} 0.98 & \text{if } 0 < w \leq 1 \\ 1.19 & \text{if } 1 < w \leq 2 \\ 1.40 & \text{if } 2 < w \leq 3 \\ 1.61 & \text{if } 3 < w \leq 4 \\ \vdots & \end{cases}$$

The graph is shown in Figure 18. You can see why functions similar to this one are called **step functions**—they jump from one value to the next. Such functions will be studied in Chapter 2.

Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect to the y -axis (see Figure 19). This means that if we have plotted the graph of f for $x \geq 0$, we obtain the entire graph simply by reflecting this portion about the y -axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

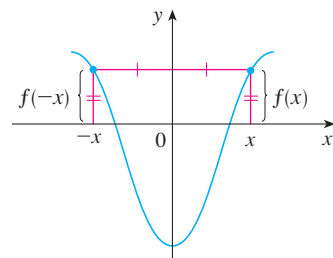


FIGURE 19

An even function

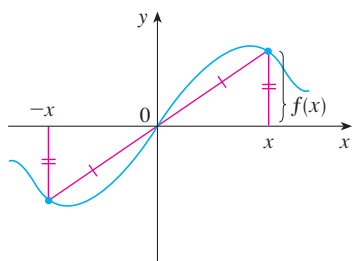


FIGURE 20
An odd function

The graph of an odd function is symmetric about the origin (see Figure 20). If we already have the graph of f for $x \geq 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

EXAMPLE 11 Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

SOLUTION

$$\begin{aligned} \text{(a)} \quad f(-x) &= (-x)^5 + (-x) = (-1)^5 x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

$$\text{(b)} \quad g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

$$\text{(c)} \quad h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd. ■

The graphs of the functions in Example 11 are shown in Figure 21. Notice that the graph of h is symmetric neither about the y -axis nor about the origin.

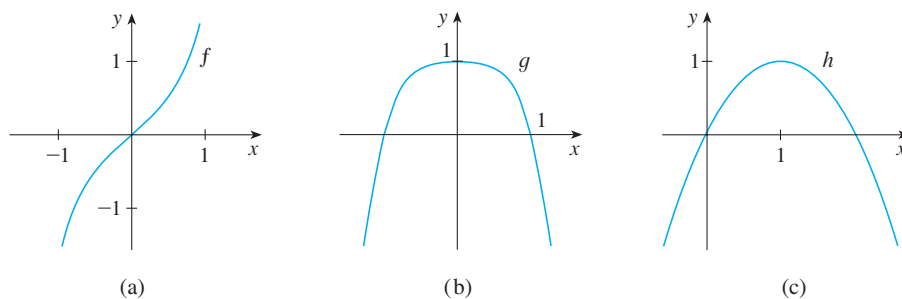


FIGURE 21

■ Increasing and Decreasing Functions

The graph shown in Figure 22 rises from A to B , falls from B to C , and rises again from C to D . The function f is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$. Notice that if x_1 and x_2 are any two numbers between a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$. We use this as the defining property of an increasing function.

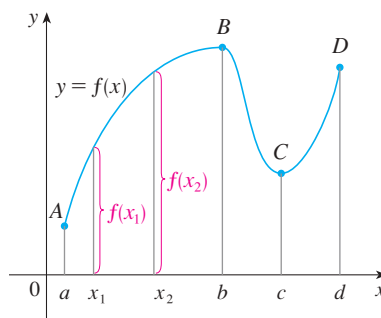


FIGURE 22

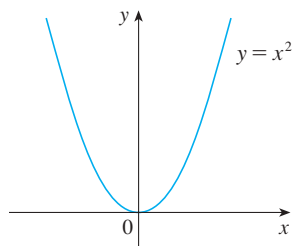


FIGURE 23

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

You can see from Figure 23 that the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

1.1 EXERCISES

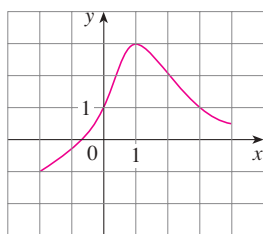
1. If $f(x) = x + \sqrt{2 - x}$ and $g(u) = u + \sqrt{2 - u}$, is it true that $f = g$?

2. If

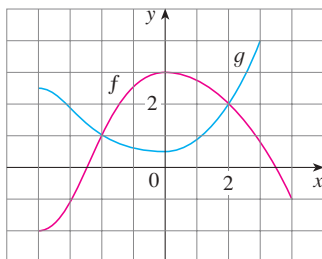
$$f(x) = \frac{x^2 - x}{x - 1} \quad \text{and} \quad g(x) = x$$

is it true that $f = g$?

3. The graph of a function f is given.
- State the value of $f(1)$.
 - Estimate the value of $f(-1)$.
 - For what values of x is $f(x) = 1$?
 - Estimate the value of x such that $f(x) = 0$.
 - State the domain and range of f .
 - On what interval is f increasing?



4. The graphs of f and g are given.



- State the values of $f(-4)$ and $g(3)$.
- For what values of x is $f(x) = g(x)$?

- Estimate the solution of the equation $f(x) = -1$.

- On what interval is f decreasing?

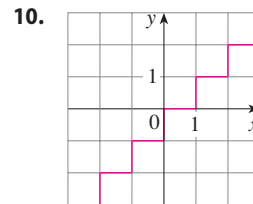
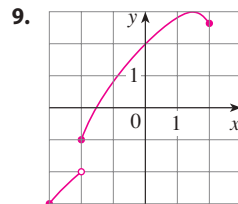
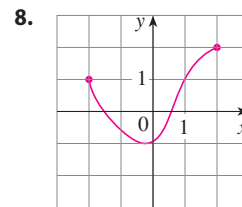
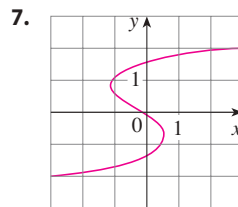
- State the domain and range of f .

- State the domain and range of g .

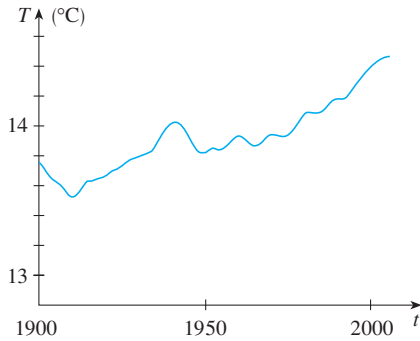
5. Figure 1 was recorded by an instrument operated by the California Department of Mines and Geology at the University Hospital of the University of Southern California in Los Angeles. Use it to estimate the range of the vertical ground acceleration function at USC during the Northridge earthquake.

6. In this section we discussed examples of ordinary, everyday functions: Population is a function of time, postage cost is a function of weight, water temperature is a function of time. Give three other examples of functions from everyday life that are described verbally. What can you say about the domain and range of each of your functions? If possible, sketch a rough graph of each function.

- 7–10** Determine whether the curve is the graph of a function of x . If it is, state the domain and range of the function.

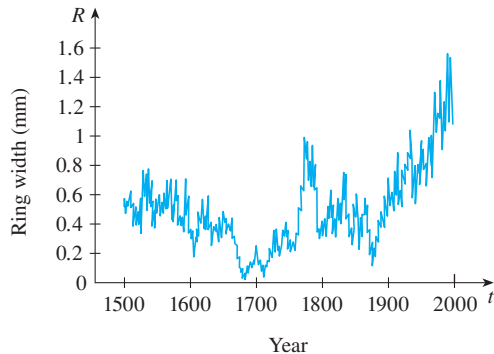


11. Shown is a graph of the global average temperature T during the 20th century. Estimate the following.
- The global average temperature in 1950
 - The year when the average temperature was 14.2°C
 - The year when the temperature was smallest; the year it was largest
 - The range of T



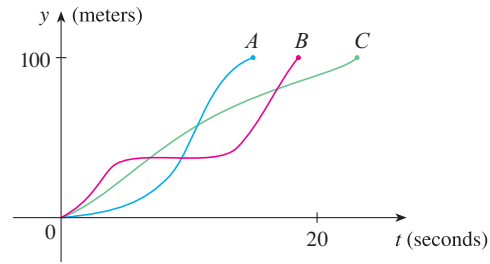
Source: Adapted from *Globe and Mail* [Toronto], 5 Dec. 2009. Print.

12. Trees grow faster and form wider rings in warm years and grow more slowly and form narrower rings in cooler years. The figure shows ring widths of a Siberian pine from 1500 to 2000.
- What is the range of the ring width function?
 - What does the graph tend to say about the temperature of the earth? Does the graph reflect the volcanic eruptions of the mid-19th century?

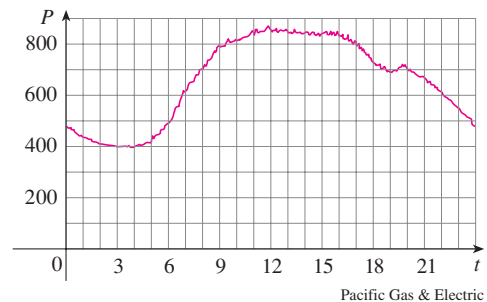


Source: Adapted from G. Jacoby et al., "Mongolian Tree Rings and 20th-Century Warming," *Science* 273 (1996): 771–73.

13. You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table. Describe how the temperature of the water changes as time passes. Then sketch a rough graph of the temperature of the water as a function of the elapsed time.
14. Three runners compete in a 100-meter race. The graph depicts the distance run as a function of time for each runner. Describe in words what the graph tells you about this race. Who won the race? Did each runner finish the race?



15. The graph shows the power consumption for a day in September in San Francisco. (P is measured in megawatts; t is measured in hours starting at midnight.)
- What was the power consumption at 6 AM? At 6 PM?
 - When was the power consumption the lowest? When was it the highest? Do these times seem reasonable?



16. Sketch a rough graph of the number of hours of daylight as a function of the time of year.
17. Sketch a rough graph of the outdoor temperature as a function of time during a typical spring day.
18. Sketch a rough graph of the market value of a new car as a function of time for a period of 20 years. Assume the car is well maintained.
19. Sketch the graph of the amount of a particular brand of coffee sold by a store as a function of the price of the coffee.
20. You place a frozen pie in an oven and bake it for an hour. Then you take it out and let it cool before eating it. Describe how the temperature of the pie changes as time passes. Then sketch a rough graph of the temperature of the pie as a function of time.
21. A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period.
22. An airplane takes off from an airport and lands an hour later at another airport, 400 miles away. If t represents the time in minutes since the plane has left the terminal building, let $x(t)$ be the horizontal distance traveled and $y(t)$ be the altitude of the plane.
- Sketch a possible graph of $x(t)$.
 - Sketch a possible graph of $y(t)$.

- (c) Sketch a possible graph of the ground speed.
 (d) Sketch a possible graph of the vertical velocity.

- 23.** Temperature readings T (in $^{\circ}\text{F}$) were recorded every two hours from midnight to 2:00 PM in Atlanta on June 4, 2013. The time t was measured in hours from midnight.

t	0	2	4	6	8	10	12	14
T	74	69	68	66	70	78	82	86

- (a) Use the readings to sketch a rough graph of T as a function of t .
 (b) Use your graph to estimate the temperature at 9:00 AM.
- 24.** Researchers measured the blood alcohol concentration (BAC) of eight adult male subjects after rapid consumption of 30 mL of ethanol (corresponding to two standard alcoholic drinks). The table shows the data they obtained by averaging the BAC (in g/dL) of the eight men.
- (a) Use the readings to sketch the graph of the BAC as a function of t .
 (b) Use your graph to describe how the effect of alcohol varies with time.

t (hours)	BAC	t (hours)	BAC
0	0	1.75	0.022
0.2	0.025	2.0	0.018
0.5	0.041	2.25	0.015
0.75	0.040	2.5	0.012
1.0	0.033	3.0	0.007
1.25	0.029	3.5	0.003
1.5	0.024	4.0	0.001

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

- 25.** If $f(x) = 3x^2 - x + 2$, find $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(a + 1)$, $2f(a)$, $f(2a)$, $f(a^2)$, $[f(a)]^2$, and $f(a + h)$.
- 26.** A spherical balloon with radius r inches has volume $V(r) = \frac{4}{3}\pi r^3$. Find a function that represents the amount of air required to inflate the balloon from a radius of r inches to a radius of $r + 1$ inches.

- 27–30** Evaluate the difference quotient for the given function. Simplify your answer.

27. $f(x) = 4 + 3x - x^2$, $\frac{f(3 + h) - f(3)}{h}$

28. $f(x) = x^3$, $\frac{f(a + h) - f(a)}{h}$

29. $f(x) = \frac{1}{x}$, $\frac{f(x) - f(a)}{x - a}$

30. $f(x) = \frac{x + 3}{x + 1}$, $\frac{f(x) - f(1)}{x - 1}$

- 31–37** Find the domain of the function.

31. $f(x) = \frac{x + 4}{x^2 - 9}$

32. $f(x) = \frac{2x^3 - 5}{x^2 + x - 6}$

33. $f(t) = \sqrt[3]{2t - 1}$

34. $g(t) = \sqrt{3 - t} - \sqrt{2 + t}$

35. $h(x) = \frac{1}{\sqrt[4]{x^2 - 5x}}$

36. $f(u) = \frac{u + 1}{1 + \frac{1}{u + 1}}$

37. $F(p) = \sqrt{2 - \sqrt{p}}$

- 38.** Find the domain and range and sketch the graph of the function $h(x) = \sqrt{4 - x^2}$.

- 39–40** Find the domain and sketch the graph of the function.

39. $f(x) = 1.6x - 2.4$

40. $g(t) = \frac{t^2 - 1}{t + 1}$

- 41–44** Evaluate $f(-3)$, $f(0)$, and $f(2)$ for the piecewise defined function. Then sketch the graph of the function.

41. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 1 - x & \text{if } x \geq 0 \end{cases}$

42. $f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x < 2 \\ 2x - 5 & \text{if } x \geq 2 \end{cases}$

43. $f(x) = \begin{cases} x + 1 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

44. $f(x) = \begin{cases} -1 & \text{if } x \leq 1 \\ 7 - 2x & \text{if } x > 1 \end{cases}$

- 45–50** Sketch the graph of the function.

45. $f(x) = x + |x|$

46. $f(x) = |x + 2|$

47. $g(t) = |1 - 3t|$

48. $h(t) = |t| + |t + 1|$

49. $f(x) = \begin{cases} |x| & \text{if } |x| \leq 1 \\ 1 & \text{if } |x| > 1 \end{cases}$

50. $g(x) = ||x| - 1|$

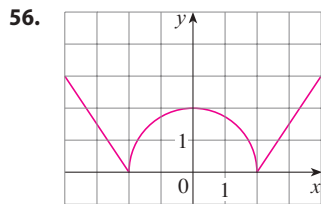
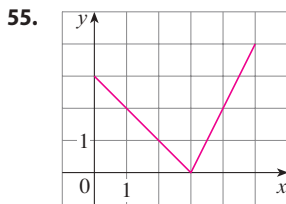
- 51–56** Find an expression for the function whose graph is the given curve.

51. The line segment joining the points $(1, -3)$ and $(5, 7)$

52. The line segment joining the points $(-5, 10)$ and $(7, -10)$

53. The bottom half of the parabola $x + (y - 1)^2 = 0$

54. The top half of the circle $x^2 + (y - 2)^2 = 4$



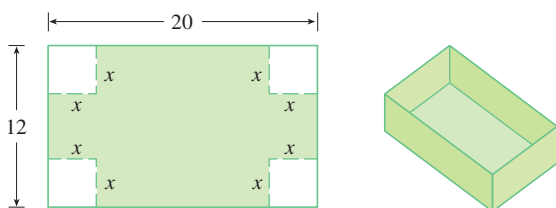
57–61 Find a formula for the described function and state its domain.

- 57.** A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.
- 58.** A rectangle has area 16 m^2 . Express the perimeter of the rectangle as a function of the length of one of its sides.
- 59.** Express the area of an equilateral triangle as a function of the length of a side.
- 60.** A closed rectangular box with volume 8 ft^3 has length twice the width. Express the height of the box as a function of the width.
- 61.** An open rectangular box with volume 2 m^3 has a square base. Express the surface area of the box as a function of the length of a side of the base.

- 62.** A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as a function of the width x of the window.

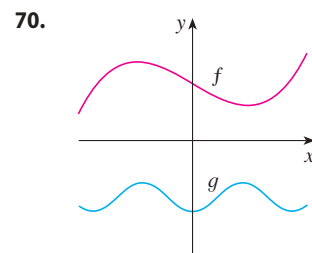
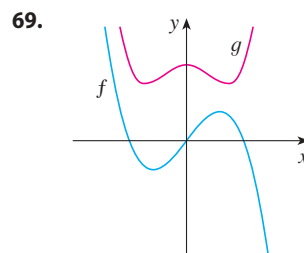


- 63.** A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x .

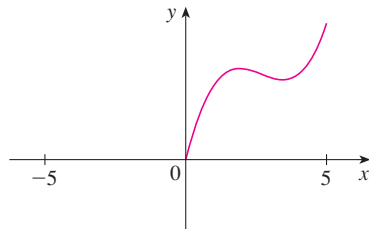


- 64.** A cell phone plan has a basic charge of \$35 a month. The plan includes 400 free minutes and charges 10 cents for each additional minute of usage. Write the monthly cost C as a function of the number x of minutes used and graph C as a function of x for $0 \leq x \leq 600$.
- 65.** In a certain state the maximum speed permitted on freeways is 65 mi/h and the minimum speed is 40 mi/h. The fine for violating these limits is \$15 for every mile per hour above the maximum speed or below the minimum speed. Express the amount of the fine F as a function of the driving speed x and graph $F(x)$ for $0 \leq x \leq 100$.
- 66.** An electricity company charges its customers a base rate of \$10 a month, plus 6 cents per kilowatt-hour (kWh) for the first 1200 kWh and 7 cents per kWh for all usage over 1200 kWh. Express the monthly cost E as a function of the amount x of electricity used. Then graph the function E for $0 \leq x \leq 2000$.
- 67.** In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.
- Sketch the graph of the tax rate R as a function of the income I .
 - How much tax is assessed on an income of \$14,000? On \$26,000?
 - Sketch the graph of the total assessed tax T as a function of the income I .
- 68.** The functions in Example 10 and Exercise 67 are called *step functions* because their graphs look like stairs. Give two other examples of step functions that arise in everyday life.

69–70 Graphs of f and g are shown. Decide whether each function is even, odd, or neither. Explain your reasoning.



- 71.** (a) If the point $(5, 3)$ is on the graph of an even function, what other point must also be on the graph?
 (b) If the point $(5, 3)$ is on the graph of an odd function, what other point must also be on the graph?
- 72.** A function f has domain $[-5, 5]$ and a portion of its graph is shown.
- Complete the graph of f if it is known that f is even.
 - Complete the graph of f if it is known that f is odd.



73–78 Determine whether f is even, odd, or neither. If you have a graphing calculator, use it to check your answer visually.

73. $f(x) = \frac{x}{x^2 + 1}$

74. $f(x) = \frac{x^2}{x^4 + 1}$

75. $f(x) = \frac{x}{x + 1}$

76. $f(x) = x|x|$

77. $f(x) = 1 + 3x^2 - x^4$

78. $f(x) = 1 + 3x^3 - x^5$

79. If f and g are both even functions, is $f + g$ even? If f and g are both odd functions, is $f + g$ odd? What if f is even and g is odd? Justify your answers.

80. If f and g are both even functions, is the product fg even? If f and g are both odd functions, is fg odd? What if f is even and g is odd? Justify your answers.

1.2 Mathematical Models: A Catalog of Essential Functions

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emission reductions. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Figure 1 illustrates the process of mathematical modeling. Given a real-world problem, our first task is to formulate a mathematical model by identifying and naming the independent and dependent variables and making assumptions that simplify the phenomenon enough to make it mathematically tractable. We use our knowledge of the physical situation and our mathematical skills to obtain equations that relate the variables. In situations where there is no physical law to guide us, we may need to collect data (either from a library or the Internet or by conducting our own experiments) and examine the data in the form of a table in order to discern patterns. From this numerical representation of a function we may wish to obtain a graphical representation by plotting the data. The graph might even suggest a suitable algebraic formula in some cases.

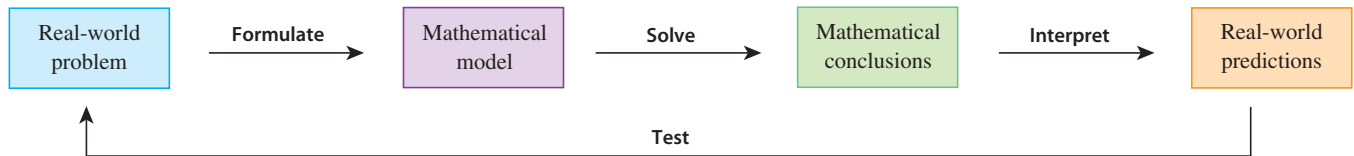


FIGURE 1
The modeling process

The second stage is to apply the mathematics that we know (such as the calculus that will be developed throughout this book) to the mathematical model that we have formulated in order to derive mathematical conclusions. Then, in the third stage, we take those mathematical conclusions and interpret them as information about the original real-world phenomenon by way of offering explanations or making predictions. The final step is to test our predictions by checking against new real data. If the predictions don't compare well with reality, we need to refine our model or to formulate a new model and start the cycle again.

A mathematical model is never a completely accurate representation of a physical situation—it is an *idealization*. A good model simplifies reality enough to permit math-

ematical calculations but is accurate enough to provide valuable conclusions. It is important to realize the limitations of the model. In the end, Mother Nature has the final say.

There are many different types of functions that can be used to model relationships observed in the real world. In what follows, we discuss the behavior and graphs of these functions and give examples of situations appropriately modeled by such functions.

Linear Models

The coordinate geometry of lines is reviewed in Appendix B.

When we say that y is a **linear function** of x , we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = mx + b$$

where m is the slope of the line and b is the y -intercept.

A characteristic feature of linear functions is that they grow at a constant rate. For instance, Figure 2 shows a graph of the linear function $f(x) = 3x - 2$ and a table of sample values. Notice that whenever x increases by 0.1, the value of $f(x)$ increases by 0.3. So $f(x)$ increases three times as fast as x . Thus the slope of the graph of $y = 3x - 2$, namely 3, can be interpreted as the rate of change of y with respect to x .

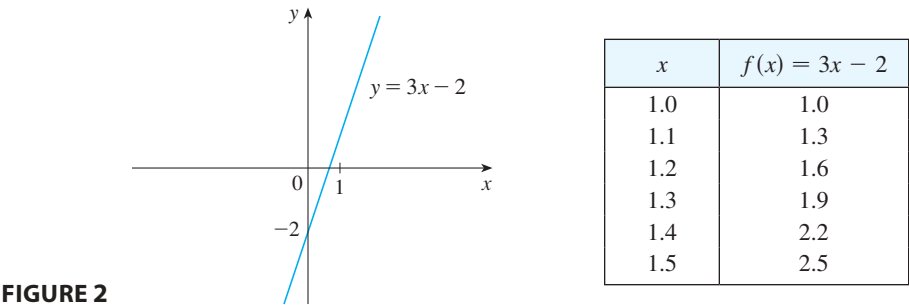


FIGURE 2

EXAMPLE 1

- (a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C , express the temperature T (in $^{\circ}\text{C}$) as a function of the height h (in kilometers), assuming that a linear model is appropriate.
- (b) Draw the graph of the function in part (a). What does the slope represent?
- (c) What is the temperature at a height of 2.5 km?

SOLUTION

- (a) Because we are assuming that T is a linear function of h , we can write

$$T = mh + b$$

We are given that $T = 20$ when $h = 0$, so

$$20 = m \cdot 0 + b = b$$

In other words, the y -intercept is $b = 20$.

We are also given that $T = 10$ when $h = 1$, so

$$10 = m \cdot 1 + 20$$

The slope of the line is therefore $m = 10 - 20 = -10$ and the required linear function is

$$T = -10h + 20$$

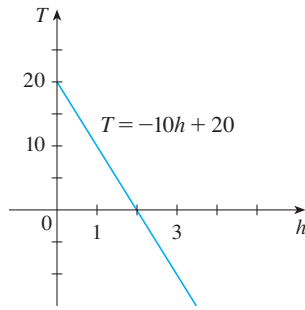


FIGURE 3

(b) The graph is sketched in Figure 3. The slope is $m = -10^\circ\text{C}/\text{km}$, and this represents the rate of change of temperature with respect to height.

(c) At a height of $h = 2.5$ km, the temperature is

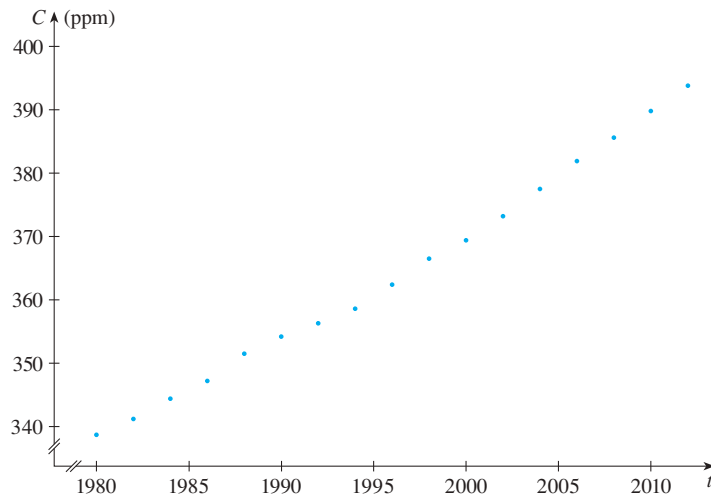
$$T = -10(2.5) + 20 = -5^\circ\text{C}$$

If there is no physical law or principle to help us formulate a model, we construct an **empirical model**, which is based entirely on collected data. We seek a curve that “fits” the data in the sense that it captures the basic trend of the data points.

EXAMPLE 2 Table 1 lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2012. Use the data in Table 1 to find a model for the carbon dioxide level.

SOLUTION We use the data in Table 1 to make the scatter plot in Figure 4, where t represents time (in years) and C represents the CO_2 level (in parts per million, ppm).

Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	1998	366.5
1982	341.2	2000	369.4
1984	344.4	2002	373.2
1986	347.2	2004	377.5
1988	351.5	2006	381.9
1990	354.2	2008	385.6
1992	356.3	2010	389.9
1994	358.6	2012	393.8
1996	362.4		

FIGURE 4 Scatter plot for the average CO_2 level

Notice that the data points appear to lie close to a straight line, so it's natural to choose a linear model in this case. But there are many possible lines that approximate these data points, so which one should we use? One possibility is the line that passes through the first and last data points. The slope of this line is

$$\frac{393.8 - 338.7}{2012 - 1980} = \frac{55.1}{32} = 1.721875 \approx 1.722$$

We write its equation as

$$C - 338.7 = 1.722(t - 1980)$$

or

1

$$C = 1.722t - 3070.86$$

Equation 1 gives one possible linear model for the carbon dioxide level; it is graphed in Figure 5.

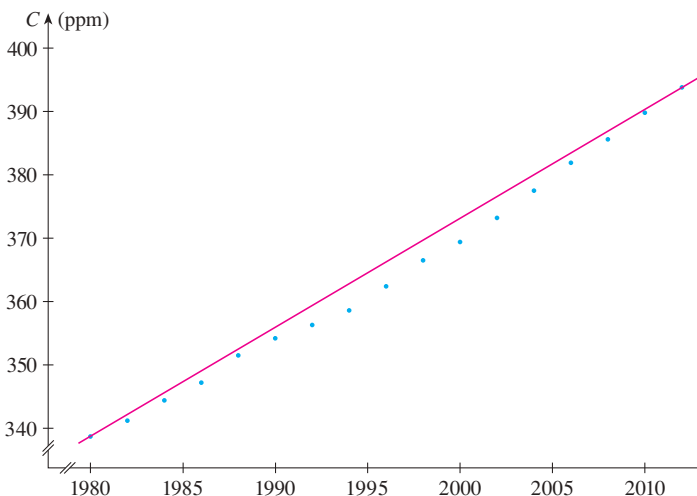


FIGURE 5
Linear model through first
and last data points

A computer or graphing calculator finds the regression line by the method of **least squares**, which is to minimize the sum of the squares of the vertical distances between the data points and the line. The details are explained in Section 14.7.

Notice that our model gives values higher than most of the actual CO_2 levels. A better linear model is obtained by a procedure from statistics called *linear regression*. If we use a graphing calculator, we enter the data from Table 1 into the data editor and choose the linear regression command. (With Maple we use the `fit[leastsquare]` command in the stats package; with Mathematica we use the `Fit` command.) The machine gives the slope and y-intercept of the regression line as

$$m = 1.71262 \quad b = -3054.14$$

So our least squares model for the CO_2 level is

$$\boxed{2} \quad C = 1.71262t - 3054.14$$

In Figure 6 we graph the regression line as well as the data points. Comparing with Figure 5, we see that it gives a better fit than our previous linear model.

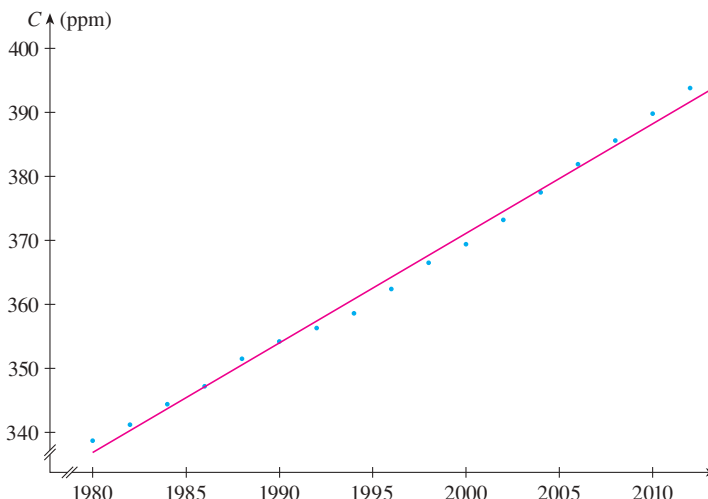


FIGURE 6
The regression line

EXAMPLE 3 Use the linear model given by Equation 2 to estimate the average CO₂ level for 1987 and to predict the level for the year 2020. According to this model, when will the CO₂ level exceed 420 parts per million?

SOLUTION Using Equation 2 with $t = 1987$, we estimate that the average CO₂ level in 1987 was

$$C(1987) = (1.71262)(1987) - 3054.14 \approx 348.84$$

This is an example of *interpolation* because we have estimated a value *between* observed values. (In fact, the Mauna Loa Observatory reported that the average CO₂ level in 1987 was 348.93 ppm, so our estimate is quite accurate.)

With $t = 2020$, we get

$$C(2020) = (1.71262)(2020) - 3054.14 \approx 405.35$$

So we predict that the average CO₂ level in the year 2020 will be 405.4 ppm. This is an example of *extrapolation* because we have predicted a value *outside* the time frame of observations. Consequently, we are far less certain about the accuracy of our prediction.

Using Equation 2, we see that the CO₂ level exceeds 420 ppm when

$$1.71262t - 3054.14 > 420$$

Solving this inequality, we get

$$t > \frac{3474.14}{1.71262} \approx 2028.55$$

We therefore predict that the CO₂ level will exceed 420 ppm by the year 2029. This prediction is risky because it involves a time quite remote from our observations. In fact, we see from Figure 6 that the trend has been for CO₂ levels to increase rather more rapidly in recent years, so the level might exceed 420 ppm well before 2029. ■

Polynomials

A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \dots, a_n$ are constants called the **coefficients** of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is n . For example, the function

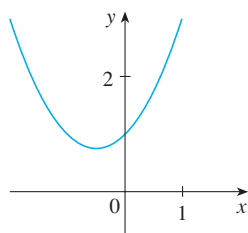
$$P(x) = 2x^6 - x^4 + \frac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

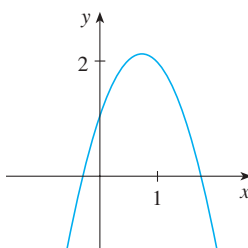
A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function. A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**. Its graph is always a parabola obtained by shifting the parabola $y = ax^2$, as we will see in the next section. The parabola opens upward if $a > 0$ and downward if $a < 0$. (See Figure 7.)

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \quad a \neq 0$$



(a) $y = x^2 + x + 1$



(b) $y = -2x^2 + 3x + 1$

FIGURE 7

The graphs of quadratic functions are parabolas.

and is called a **cubic function**. Figure 8 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c). We will see later why the graphs have these shapes.

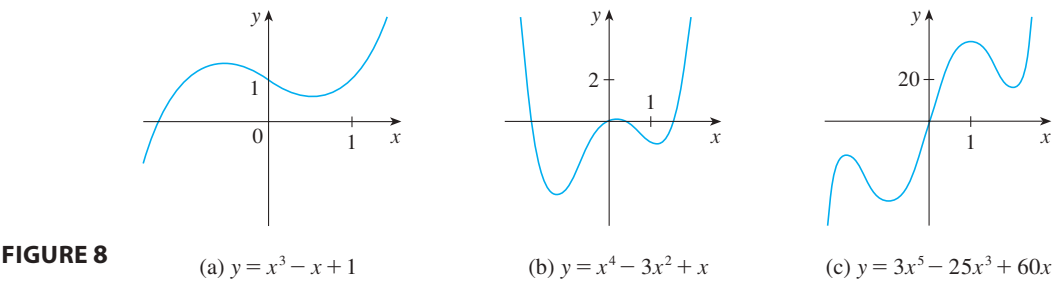


FIGURE 8

Polynomials are commonly used to model various quantities that occur in the natural and social sciences. For instance, in Section 2.7 we will explain why economists often use a polynomial $P(x)$ to represent the cost of producing x units of a commodity. In the following example we use a quadratic function to model the fall of a ball.

Table 2	
Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

EXAMPLE 4 A ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground, and its height h above the ground is recorded at 1-second intervals in Table 2. Find a model to fit the data and use the model to predict the time at which the ball hits the ground.

SOLUTION We draw a scatter plot of the data in Figure 9 and observe that a linear model is inappropriate. But it looks as if the data points might lie on a parabola, so we try a quadratic model instead. Using a graphing calculator or computer algebra system (which uses the least squares method), we obtain the following quadratic model:

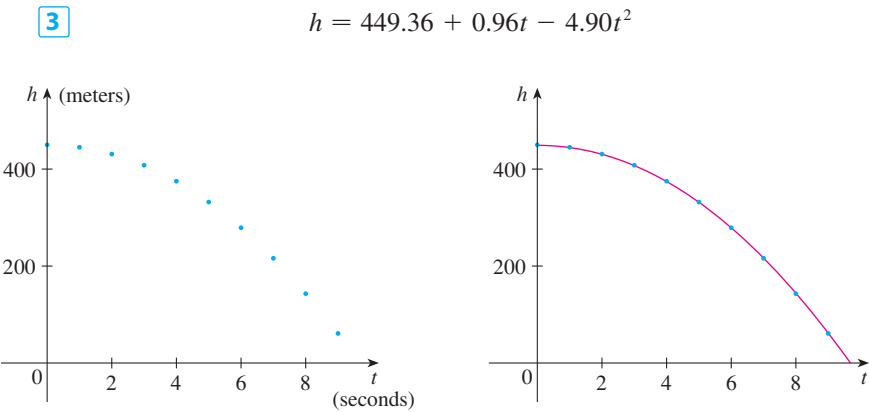


FIGURE 9
Scatter plot for a falling ball

FIGURE 10
Quadratic model for a falling ball

In Figure 10 we plot the graph of Equation 3 together with the data points and see that the quadratic model gives a very good fit. The ball hits the ground when $h = 0$, so we solve the quadratic equation

$$-4.90t^2 + 0.96t + 449.36 = 0$$

The quadratic formula gives

$$t = \frac{-0.96 \pm \sqrt{(0.96)^2 - 4(-4.90)(449.36)}}{2(-4.90)}$$

The positive root is $t \approx 9.67$, so we predict that the ball will hit the ground after about 9.7 seconds. ■

■ Power Functions

A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider several cases.

(i) $a = n$, where n is a positive integer

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 11. (These are polynomials with only one term.) We already know the shape of the graphs of $y = x$ (a line through the origin with slope 1) and $y = x^2$ [a parabola, see Example 1.1.2(b)].

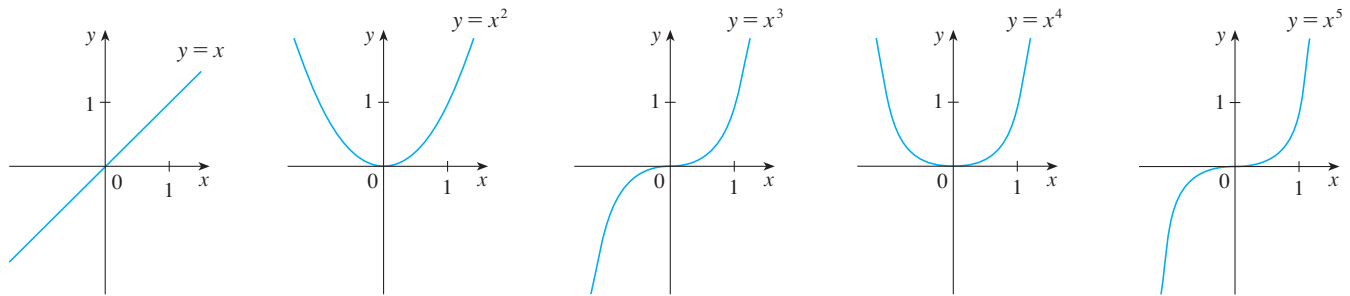


FIGURE 11 Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd. If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$. If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$. Notice from Figure 12, however, that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geq 1$. (If x is small, then x^2 is smaller, x^3 is even smaller, x^4 is smaller still, and so on.)

A **family of functions** is a collection of functions whose equations are related. Figure 12 shows two families of power functions, one with even powers and one with odd powers.

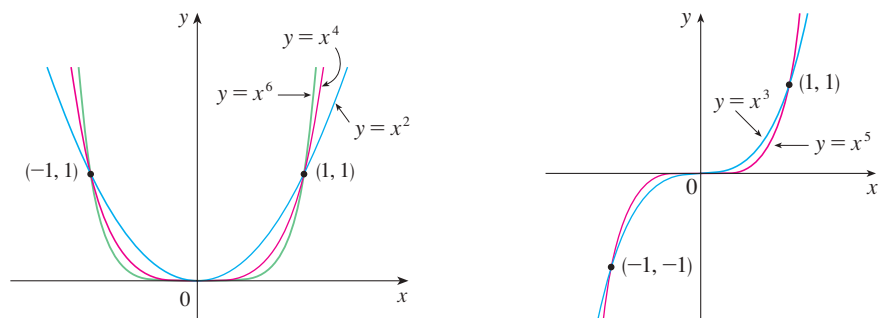


FIGURE 12

(ii) $a = 1/n$, where n is a positive integer

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For $n = 2$ it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the

parabola $x = y^2$. [See Figure 13(a).] For other even values of n , the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$. For $n = 3$ we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$ (recall that every real number has a cube root) and whose graph is shown in Figure 13(b). The graph of $y = \sqrt[n]{x}$ for n odd ($n > 3$) is similar to that of $y = \sqrt[3]{x}$.

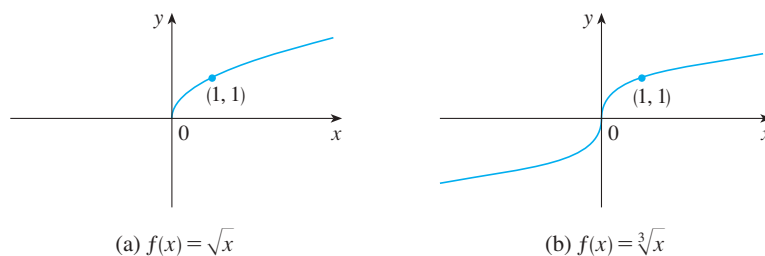


FIGURE 13
Graphs of root functions

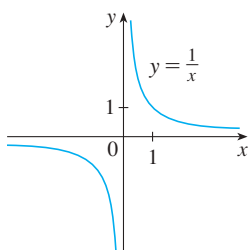


FIGURE 14
The reciprocal function

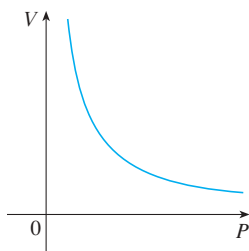


FIGURE 15
Volume as a function of pressure
at constant temperature

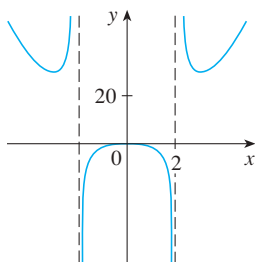


FIGURE 16
 $f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$

(iii) $a = -1$

The graph of the **reciprocal function** $f(x) = x^{-1} = 1/x$ is shown in Figure 14. Its graph has the equation $y = 1/x$, or $xy = 1$, and is a hyperbola with the coordinate axes as its asymptotes. This function arises in physics and chemistry in connection with Boyle's Law, which says that, when the temperature is constant, the volume V of a gas is inversely proportional to the pressure P :

$$V = \frac{C}{P}$$

where C is a constant. Thus the graph of V as a function of P (see Figure 15) has the same general shape as the right half of Figure 14.

Power functions are also used to model species-area relationships (Exercises 30–31), illumination as a function of distance from a light source (Exercise 29), and the period of revolution of a planet as a function of its distance from the sun (Exercise 32).

Rational Functions

A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain consists of all values of x such that $Q(x) \neq 0$. A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in Figure 14. The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 16.

Algebraic Functions

A function f is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function. Here are two more examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

When we sketch algebraic functions in Chapter 3, we will see that their graphs can assume a variety of shapes. Figure 17 illustrates some of the possibilities.

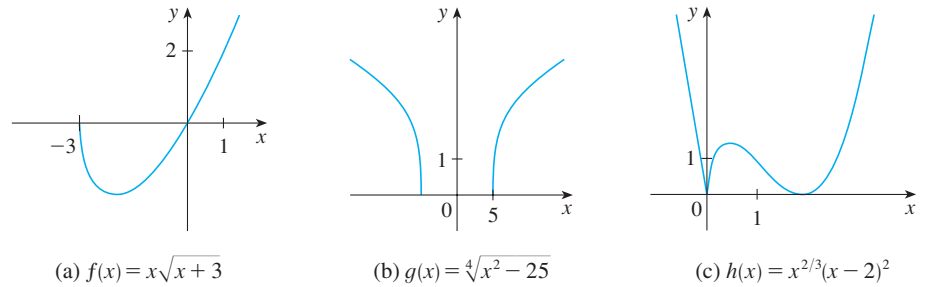


FIGURE 17

An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and $c = 3.0 \times 10^5$ km/s is the speed of light in a vacuum.

Trigonometric Functions

The Reference Pages are located at the back of the book.

Trigonometry and the trigonometric functions are reviewed on Reference Page 2 and also in Appendix D. In calculus the convention is that radian measure is always used (except when otherwise indicated). For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x . Thus the graphs of the sine and cosine functions are as shown in Figure 18.

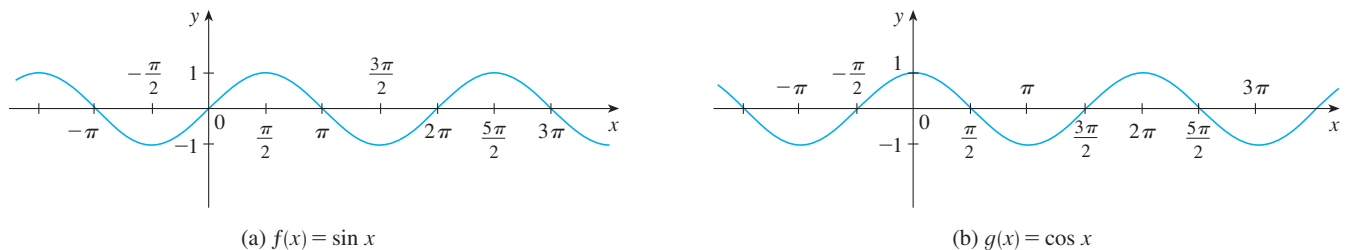


FIGURE 18

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x , we have

$$-1 \leq \sin x \leq 1 \quad -1 \leq \cos x \leq 1$$

or, in terms of absolute values,

$$|\sin x| \leq 1 \quad |\cos x| \leq 1$$

Also, the zeros of the sine function occur at the integer multiples of π ; that is,

$$\sin x = 0 \quad \text{when} \quad x = n\pi \quad n \text{ an integer}$$

An important property of the sine and cosine functions is that they are periodic functions and have period 2π . This means that, for all values of x ,

$$\sin(x + 2\pi) = \sin x \quad \cos(x + 2\pi) = \cos x$$

The periodic nature of these functions makes them suitable for modeling repetitive phenomena such as tides, vibrating springs, and sound waves. For instance, in Example 1.3.4 we will see that a reasonable model for the number of hours of daylight in Philadelphia t days after January 1 is given by the function

$$L(t) = 12 + 2.8 \sin \left[\frac{2\pi}{365}(t - 80) \right]$$

EXAMPLE 5 What is the domain of the function $f(x) = \frac{1}{1 - 2 \cos x}$?

SOLUTION This function is defined for all values of x except for those that make the denominator 0. But

$$1 - 2 \cos x = 0 \iff \cos x = \frac{1}{2} \iff x = \frac{\pi}{3} + 2n\pi \text{ or } x = \frac{5\pi}{3} + 2n\pi$$

where n is any integer (because the cosine function has period 2π). So the domain of f is the set of all real numbers except for the ones noted above. ■

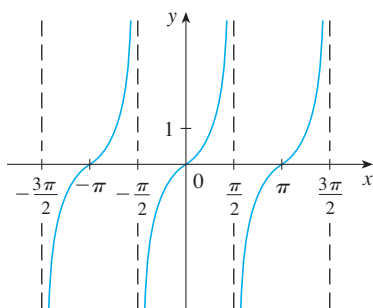


FIGURE 19
 $y = \tan x$

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

and its graph is shown in Figure 19. It is undefined whenever $\cos x = 0$, that is, when $x = \pm\pi/2, \pm3\pi/2, \dots$. Its range is $(-\infty, \infty)$. Notice that the tangent function has period π :

$$\tan(x + \pi) = \tan x \quad \text{for all } x$$

The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions. Their graphs are shown in Appendix D.

■ Exponential Functions

The **exponential functions** are the functions of the form $f(x) = b^x$, where the base b is a positive constant. The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure 20. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Exponential functions will be studied in detail in Chapter 6, and we will see that they are useful for modeling many natural phenomena, such as population growth (if $b > 1$) and radioactive decay (if $b < 1$).

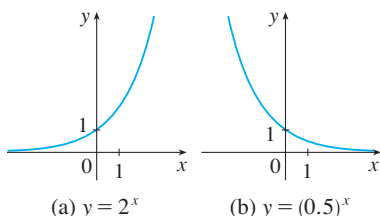


FIGURE 20

■ Logarithmic Functions

The **logarithmic functions** $f(x) = \log_b x$, where the base b is a positive constant, are the inverse functions of the exponential functions. They will be studied in Chapter 6. Figure

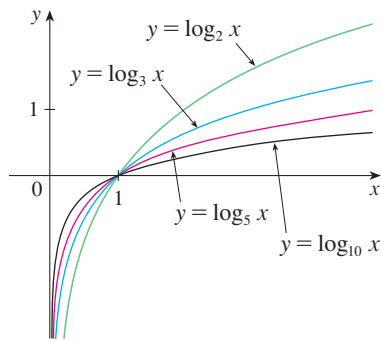


FIGURE 21

21 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

EXAMPLE 6 Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \frac{1+x}{1-\sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

SOLUTION

(a) $f(x) = 5^x$ is an exponential function. (The x is the exponent.)

(b) $g(x) = x^5$ is a power function. (The x is the base.) We could also consider it to be a polynomial of degree 5.

(c) $h(x) = \frac{1+x}{1-\sqrt{x}}$ is an algebraic function.

(d) $u(t) = 1 - t + 5t^4$ is a polynomial of degree 4. ■

1.2 EXERCISES

1–2 Classify each function as a power function, root function, polynomial (state its degree), rational function, algebraic function, trigonometric function, exponential function, or logarithmic function.

1. (a) $f(x) = \log_2 x$

(b) $g(x) = \sqrt[4]{x}$

(c) $h(x) = \frac{2x^3}{1-x^2}$

(d) $u(t) = 1 - 1.1t + 2.54t^2$

(e) $v(t) = 5^t$

(f) $w(\theta) = \sin \theta \cos^2 \theta$

2. (a) $y = \pi^x$

(b) $y = x^\pi$

(c) $y = x^2(2 - x^3)$

(d) $y = \tan t - \cos t$

(e) $y = \frac{s}{1+s}$

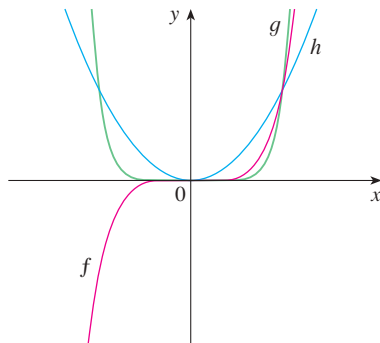
(f) $y = \frac{\sqrt{x^3-1}}{1+\sqrt[3]{x}}$

3–4 Match each equation with its graph. Explain your choices. (Don't use a computer or graphing calculator.)

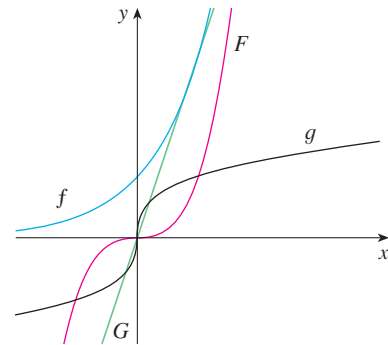
3. (a) $y = x^2$

(b) $y = x^5$

(c) $y = x^8$



4. (a) $y = 3x$ (b) $y = 3^x$ (c) $y = x^3$ (d) $y = \sqrt[3]{x}$



5–6 Find the domain of the function.

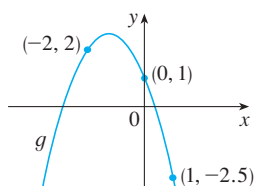
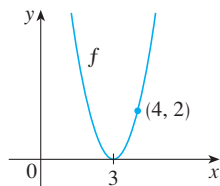
5. $f(x) = \frac{\cos x}{1 - \sin x}$

6. $g(x) = \frac{1}{1 - \tan x}$

7. (a) Find an equation for the family of linear functions with slope 2 and sketch several members of the family.
 (b) Find an equation for the family of linear functions such that $f(2) = 1$ and sketch several members of the family.
 (c) Which function belongs to both families?

8. What do all members of the family of linear functions $f(x) = 1 + m(x + 3)$ have in common? Sketch several members of the family.

9. What do all members of the family of linear functions $f(x) = c - x$ have in common? Sketch several members of the family.
10. Find expressions for the quadratic functions whose graphs are shown.

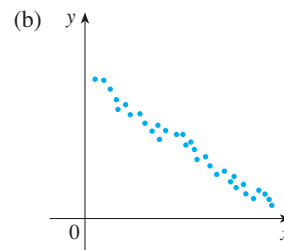
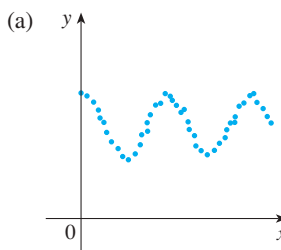


11. Find an expression for a cubic function f if $f(1) = 6$ and $f(-1) = f(0) = f(2) = 0$.
12. Recent studies indicate that the average surface temperature of the earth has been rising steadily. Some scientists have modeled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in $^{\circ}\text{C}$ and t represents years since 1900.
- What do the slope and T -intercept represent?
 - Use the equation to predict the average global surface temperature in 2100.
13. If the recommended adult dosage for a drug is D (in mg), then to determine the appropriate dosage c for a child of age a , pharmacists use the equation $c = 0.0417D(a + 1)$. Suppose the dosage for an adult is 200 mg.
- Find the slope of the graph of c . What does it represent?
 - What is the dosage for a newborn?
14. The manager of a weekend flea market knows from past experience that if he charges x dollars for a rental space at the market, then the number y of spaces he can rent is given by the equation $y = 200 - 4x$.
- Sketch a graph of this linear function. (Remember that the rental charge per space and the number of spaces rented can't be negative quantities.)
 - What do the slope, the y -intercept, and the x -intercept of the graph represent?
15. The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the linear function $F = \frac{9}{5}C + 32$.
- Sketch a graph of this function.
 - What is the slope of the graph and what does it represent? What is the F -intercept and what does it represent?
16. Jason leaves Detroit at 2:00 PM and drives at a constant speed west along I-94. He passes Ann Arbor, 40 mi from Detroit, at 2:50 PM.
- Express the distance traveled in terms of the time elapsed.
 - Draw the graph of the equation in part (a).
 - What is the slope of this line? What does it represent?

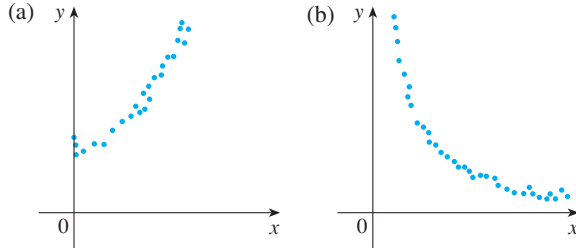
17. Biologists have noticed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 113 chirps per minute at 70°F and 173 chirps per minute at 80°F .
- Find a linear equation that models the temperature T as a function of the number of chirps per minute N .
 - What is the slope of the graph? What does it represent?
 - If the crickets are chirping at 150 chirps per minute, estimate the temperature.
18. The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day and \$4800 to produce 300 chairs in one day.
- Express the cost as a function of the number of chairs produced, assuming that it is linear. Then sketch the graph.
 - What is the slope of the graph and what does it represent?
 - What is the y -intercept of the graph and what does it represent?
19. At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 lb/in². Below the surface, the water pressure increases by 4.34 lb/in² for every 10 ft of descent.
- Express the water pressure as a function of the depth below the ocean surface.
 - At what depth is the pressure 100 lb/in²?
20. The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May it cost her \$380 to drive 480 mi and in June it cost her \$460 to drive 800 mi.
- Express the monthly cost C as a function of the distance driven d , assuming that a linear relationship gives a suitable model.
 - Use part (a) to predict the cost of driving 1500 miles per month.
 - Draw the graph of the linear function. What does the slope represent?
 - What does the C -intercept represent?
 - Why does a linear function give a suitable model in this situation?

21–22 For each scatter plot, decide what type of function you might choose as a model for the data. Explain your choices.

21.



22.



23. The table shows (lifetime) peptic ulcer rates (per 100 population) for various family incomes as reported by the National Health Interview Survey.

Income	Ulcer rate (per 100 population)
\$4,000	14.1
\$6,000	13.0
\$8,000	13.4
\$12,000	12.5
\$16,000	12.0
\$20,000	12.4
\$30,000	10.5
\$45,000	9.4
\$60,000	8.2

- Make a scatter plot of these data and decide whether a linear model is appropriate.
- Find and graph a linear model using the first and last data points.
- Find and graph the least squares regression line.
- Use the linear model in part (c) to estimate the ulcer rate for an income of \$25,000.
- According to the model, how likely is someone with an income of \$80,000 to suffer from peptic ulcers?
- Do you think it would be reasonable to apply the model to someone with an income of \$200,000?

24. Biologists have observed that the chirping rate of crickets of a certain species appears to be related to temperature. The table shows the chirping rates for various temperatures.

- Make a scatter plot of the data.
- Find and graph the regression line.
- Use the linear model in part (b) to estimate the chirping rate at 100°F.

Temperature (°F)	Chirping rate (chirps/min)	Temperature (°F)	Chirping rate (chirps/min)
50	20	75	140
55	46	80	173
60	79	85	198
65	91	90	211
70	113		

25. Anthropologists use a linear model that relates human femur (thighbone) length to height. The model allows an anthropologist to determine the height of an individual when only a partial skeleton (including the femur) is found. Here we find the model by analyzing the data on femur length and height for the eight males given in the following table.

- Make a scatter plot of the data.
- Find and graph the regression line that models the data.
- An anthropologist finds a human femur of length 53 cm. How tall was the person?

Femur length (cm)	Height (cm)	Femur length (cm)	Height (cm)
50.1	178.5	44.5	168.3
48.3	173.6	42.7	165.0
45.2	164.8	39.5	155.4
44.7	163.7	38.0	155.8

26. When laboratory rats are exposed to asbestos fibers, some of them develop lung tumors. The table lists the results of several experiments by different scientists.

- Find the regression line for the data.
- Make a scatter plot and graph the regression line.
Does the regression line appear to be a suitable model for the data?
- What does the y-intercept of the regression line represent?

Asbestos exposure (fibers/mL)	Percent of mice that develop lung tumors	Asbestos exposure (fibers/mL)	Percent of mice that develop lung tumors
50	2	1600	42
400	6	1800	37
500	5	2000	38
900	10	3000	50
1100	26		

27. The table shows world average daily oil consumption from 1985 to 2010 measured in thousands of barrels per day.

- Make a scatter plot and decide whether a linear model is appropriate.
- Find and graph the regression line.
- Use the linear model to estimate the oil consumption in 2002 and 2012.

Years since 1985	Thousands of barrels of oil per day
0	60,083
5	66,533
10	70,099
15	76,784
20	84,077
25	87,302

Source: US Energy Information Administration

-  **28.** The table shows average US retail residential prices of electricity from 2000 to 2012, measured in cents per kilowatt hour.
- Make a scatter plot. Is a linear model appropriate?
 - Find and graph the regression line.
 - Use your linear model from part (b) to estimate the average retail price of electricity in 2005 and 2013.

Years since 2000	Cents/kWh
0	8.24
2	8.44
4	8.95
6	10.40
8	11.26
10	11.54
12	11.58

Source: US Energy Information Administration

- 29.** Many physical quantities are connected by *inverse square laws*, that is, by power functions of the form $f(x) = kx^{-2}$. In particular, the illumination of an object by a light source is inversely proportional to the square of the distance from the source. Suppose that after dark you are in a room with just one lamp and you are trying to read a book. The light is too dim and so you move halfway to the lamp. How much brighter is the light?
- 30.** It makes sense that the larger the area of a region, the larger the number of species that inhabit the region. Many ecologists have modeled the species-area relation with a power function and, in particular, the number of species S of bats living in caves in central Mexico has been related to the surface area A of the caves by the equation $S = 0.7A^{0.3}$.
- The cave called *Misión Imposible* near Puebla, Mexico, has a surface area of $A = 60 \text{ m}^2$. How many species of bats would you expect to find in that cave?
 - If you discover that four species of bats live in a cave, estimate the area of the cave.

-  **31.** The table shows the number N of species of reptiles and amphibians inhabiting Caribbean islands and the area A of the island in square miles.
- Use a power function to model N as a function of A .
 - The Caribbean island of Dominica has area 291 mi^2 . How many species of reptiles and amphibians would you expect to find on Dominica?

Island	A	N
Saba	4	5
Montserrat	40	9
Puerto Rico	3,459	40
Jamaica	4,411	39
Hispaniola	29,418	84
Cuba	44,218	76

-  **32.** The table shows the mean (average) distances d of the planets from the sun (taking the unit of measurement to be the distance from planet Earth to the sun) and their periods T (time of revolution in years).
- Fit a power model to the data.
 - Kepler's Third Law of Planetary Motion states that "The square of the period of revolution of a planet is proportional to the cube of its mean distance from the sun." Does your model corroborate Kepler's Third Law?

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784

1.3 New Functions from Old Functions

In this section we start with the basic functions we discussed in Section 1.2 and obtain new functions by shifting, stretching, and reflecting their graphs. We also show how to combine pairs of functions by the standard arithmetic operations and by composition.

■ Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.

Let's first consider **translations**. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y -coordinate is increased by the same number c). Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x). There-

for the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted c units to the right (see Figure 1).

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

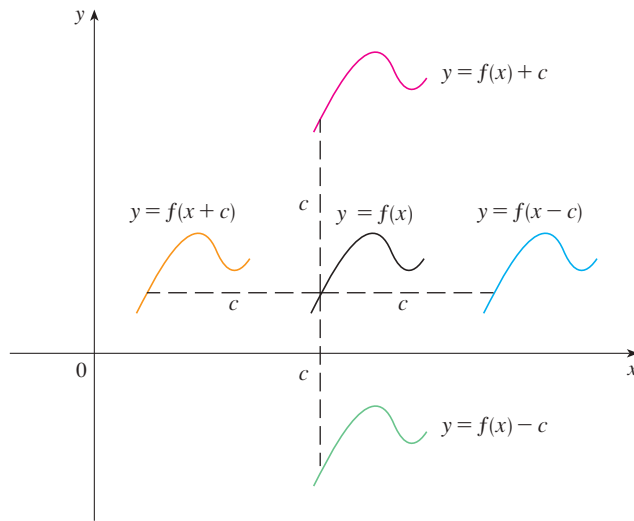


FIGURE 1 Translating the graph of f

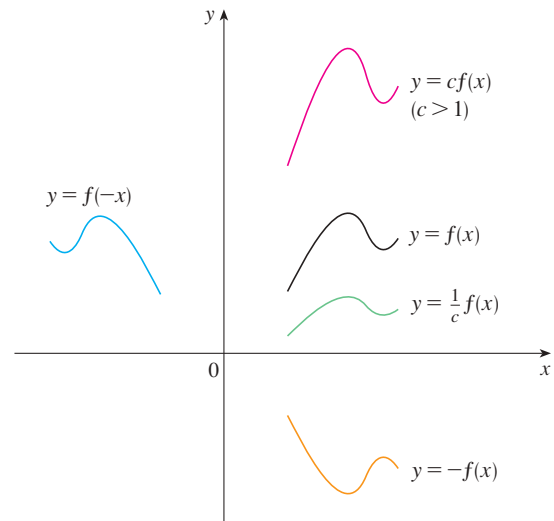


FIGURE 2 Stretching and reflecting the graph of f

Now let's consider the **stretching** and **reflecting** transformations. If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y -coordinate is multiplied by the same number c). The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x -axis because the point (x, y) is replaced by the point $(x, -y)$. (See Figure 2 and the following chart, where the results of other stretching, shrinking, and reflecting transformations are also given.)

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c

$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x -axis

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y -axis

Figure 3 illustrates these stretching transformations when applied to the cosine function with $c = 2$. For instance, in order to get the graph of $y = 2 \cos x$ we multiply the y -coordi-

nate of each point on the graph of $y = \cos x$ by 2. This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

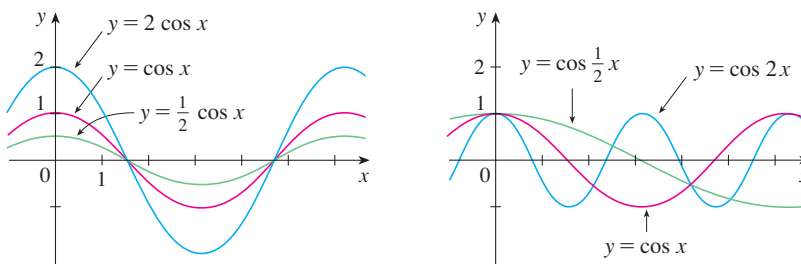


FIGURE 3

EXAMPLE 1 Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

SOLUTION The graph of the square root function $y = \sqrt{x}$, obtained from Figure 1.2.13(a), is shown in Figure 4(a). In the other parts of the figure we sketch $y = \sqrt{x} - 2$ by shifting 2 units downward, $y = \sqrt{x - 2}$ by shifting 2 units to the right, $y = -\sqrt{x}$ by reflecting about the x -axis, $y = 2\sqrt{x}$ by stretching vertically by a factor of 2, and $y = \sqrt{-x}$ by reflecting about the y -axis.

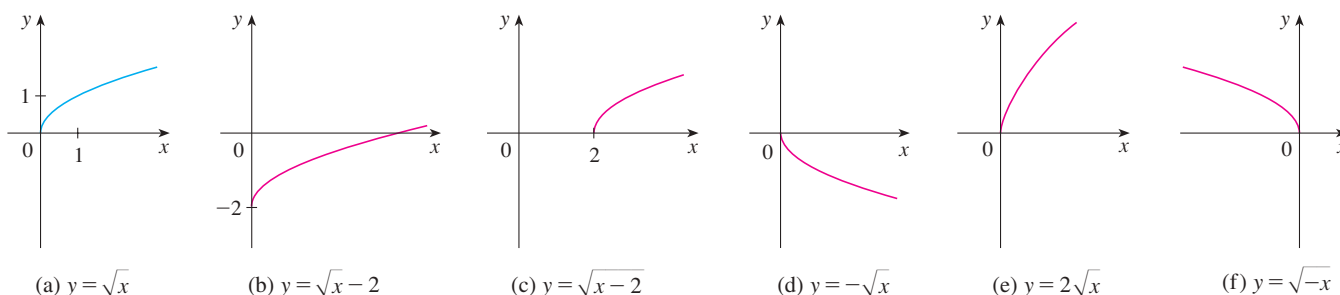


FIGURE 4

EXAMPLE 2 Sketch the graph of the function $f(x) = x^2 + 6x + 10$.

SOLUTION Completing the square, we write the equation of the graph as

$$y = x^2 + 6x + 10 = (x + 3)^2 + 1$$

This means we obtain the desired graph by starting with the parabola $y = x^2$ and shifting 3 units to the left and then 1 unit upward (see Figure 5).

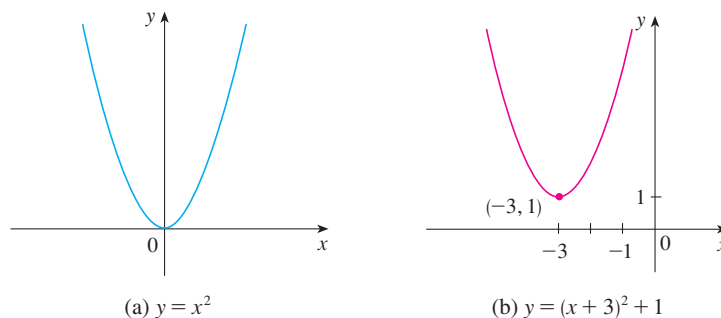


FIGURE 5

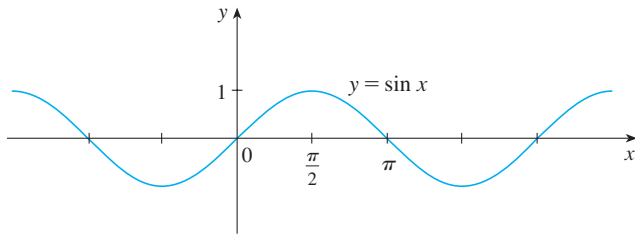
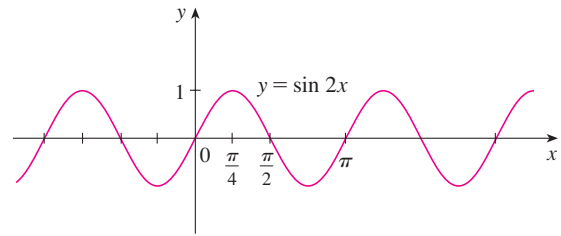
EXAMPLE 3 Sketch the graphs of the following functions.

(a) $y = \sin 2x$

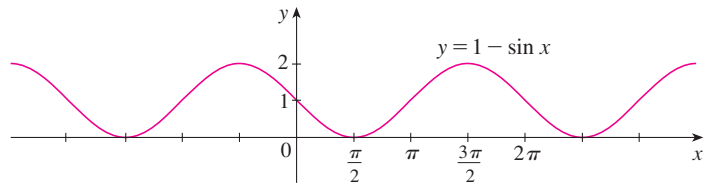
(b) $y = 1 - \sin x$

SOLUTION

(a) We obtain the graph of $y = \sin 2x$ from that of $y = \sin x$ by compressing horizontally by a factor of 2. (See Figures 6 and 7.) Thus, whereas the period of $y = \sin x$ is 2π , the period of $y = \sin 2x$ is $2\pi/2 = \pi$.

**FIGURE 6****FIGURE 7**

(b) To obtain the graph of $y = 1 - \sin x$, we again start with $y = \sin x$. We reflect about the x -axis to get the graph of $y = -\sin x$ and then we shift 1 unit upward to get $y = 1 - \sin x$. (See Figure 8.)

**FIGURE 8**

EXAMPLE 4 Figure 9 shows graphs of the number of hours of daylight as functions of the time of the year at several latitudes. Given that Philadelphia is located at approximately 40°N latitude, find a function that models the length of daylight at Philadelphia.

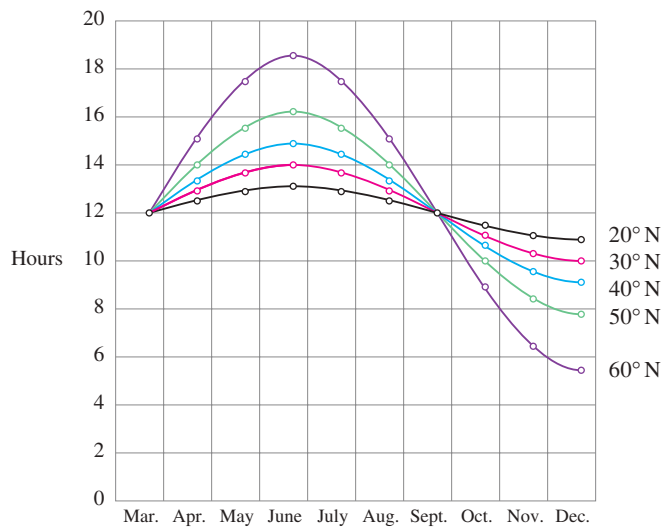


FIGURE 9
Graph of the length of daylight from
March 21 through December 21
at various latitudes

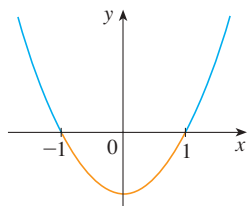
Source: Adapted from L. Harrison,
Daylight, Twilight, Darkness and Time
(New York: Silver, Burdett, 1935), 40.

SOLUTION Notice that each curve resembles a shifted and stretched sine function. By looking at the blue curve we see that, at the latitude of Philadelphia, daylight lasts about 14.8 hours on June 21 and 9.2 hours on December 21, so the amplitude of the curve (the factor by which we have to stretch the sine curve vertically) is $\frac{1}{2}(14.8 - 9.2) = 2.8$.

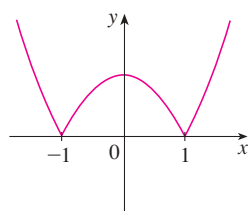
By what factor do we need to stretch the sine curve horizontally if we measure the time t in days? Because there are about 365 days in a year, the period of our model should be 365. But the period of $y = \sin t$ is 2π , so the horizontal stretching factor is $2\pi/365$.

We also notice that the curve begins its cycle on March 21, the 80th day of the year, so we have to shift the curve 80 units to the right. In addition, we shift it 12 units upward. Therefore we model the length of daylight in Philadelphia on the t th day of the year by the function

$$L(t) = 12 + 2.8 \sin \left[\frac{2\pi}{365}(t - 80) \right]$$



(a) $y = x^2 - 1$



(b) $y = |x^2 - 1|$

FIGURE 10

Another transformation of some interest is taking the *absolute value* of a function. If $y = |f(x)|$, then according to the definition of absolute value, $y = f(x)$ when $f(x) \geq 0$ and $y = -f(x)$ when $f(x) < 0$. This tells us how to get the graph of $y = |f(x)|$ from the graph of $y = f(x)$: the part of the graph that lies above the x -axis remains the same; the part that lies below the x -axis is reflected about the x -axis.

EXAMPLE 5 Sketch the graph of the function $y = |x^2 - 1|$.

SOLUTION We first graph the parabola $y = x^2 - 1$ in Figure 10(a) by shifting the parabola $y = x^2$ downward 1 unit. We see that the graph lies below the x -axis when $-1 < x < 1$, so we reflect that part of the graph about the x -axis to obtain the graph of $y = |x^2 - 1|$ in Figure 10(b).

Combinations of Functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg , and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. The sum and difference functions are defined by

$$(f + g)(x) = f(x) + g(x) \quad (f - g)(x) = f(x) - g(x)$$

If the domain of f is A and the domain of g is B , then the domain of $f + g$ is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined. For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}$ is $A \cap B = [0, 2]$.

Similarly, the product and quotient functions are defined by

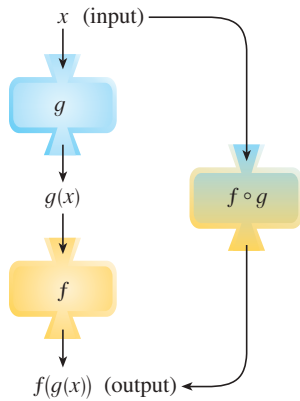
$$(fg)(x) = f(x)g(x) \quad \left(\frac{f}{g} \right)(x) = \frac{f(x)}{g(x)}$$

The domain of fg is $A \cap B$. Because we can't divide by 0, the domain of f/g is therefore $\{x \in A \cap B \mid g(x) \neq 0\}$. For instance, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $(f/g)(x) = x^2/(x - 1)$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$. Since y is a function of u and u is, in turn, a function of x , it follows that y is ultimately a function of x . We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g .

**FIGURE 11**

The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

In general, given any two functions f and g , we start with a number x in the domain of g and calculate $g(x)$. If this number $g(x)$ is in the domain of f , then we can calculate the value of $f(g(x))$. Notice that the output of one function is used as the input to the next function. The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f . It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (“ f circle g ”).

Definition Given two functions f and g , the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f . In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. Figure 11 shows how to picture $f \circ g$ in terms of machines.

EXAMPLE 6 If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

SOLUTION We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

NOTE You can see from Example 6 that, in general, $f \circ g \neq g \circ f$. Remember, the notation $f \circ g$ means that the function g is applied first and then f is applied second. In Example 6, $f \circ g$ is the function that *first* subtracts 3 and *then* squares; $g \circ f$ is the function that *first* squares and *then* subtracts 3.

EXAMPLE 7 If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find each of the following functions and their domains.

- (a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

SOLUTION

(a) $(f \circ g)(x) = f(g(x)) = f(\sqrt{2 - x}) = \sqrt{\sqrt{2 - x}} = \sqrt[4]{2 - x}$

The domain of $f \circ g$ is $\{x \mid 2 - x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2]$.

(b) $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2 - \sqrt{x}}$

For $\sqrt{x}$ to be defined we must have $x \geq 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined we must have $2 - \sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$, or $x \leq 4$. Thus we have $0 \leq x \leq 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

(c) $(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$

The domain of $f \circ f$ is $[0, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g(\sqrt{2 - x}) = \sqrt{2 - \sqrt{2 - x}}$

This expression is defined when both $2 - x \geq 0$ and $2 - \sqrt{2 - x} \geq 0$. The first inequality means $x \leq 2$, and the second is equivalent to $\sqrt{2 - x} \leq 2$, or $2 - x \leq 4$, or $x \geq -2$. Thus $-2 \leq x \leq 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$.

If $0 \leq a \leq b$, then $a^2 \leq b^2$.

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h , then g , and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

EXAMPLE 8 Find $f \circ g \circ h$ if $f(x) = x/(x + 1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

SOLUTION

$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x + 3)) \\ &= f((x + 3)^{10}) = \frac{(x + 3)^{10}}{(x + 3)^{10} + 1}\end{aligned}$$

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

EXAMPLE 9 Given $F(x) = \cos^2(x + 9)$, find functions f , g , and h such that $F = f \circ g \circ h$.

SOLUTION Since $F(x) = [\cos(x + 9)]^2$, the formula for F says: First add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 \quad g(x) = \cos x \quad f(x) = x^2$$

$$\begin{aligned}\text{Then } (f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x + 9)) = f(\cos(x + 9)) \\ &= [\cos(x + 9)]^2 = F(x)\end{aligned}$$

1.3 EXERCISES

1. Suppose the graph of f is given. Write equations for the graphs that are obtained from the graph of f as follows.

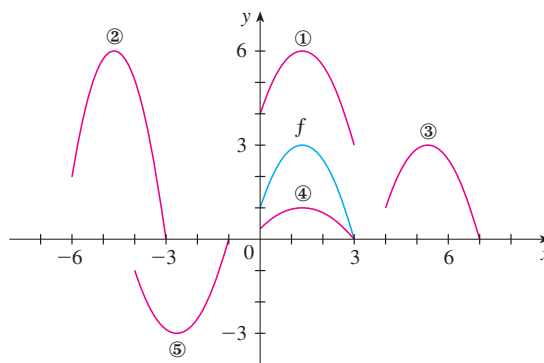
- Shift 3 units upward.
- Shift 3 units downward.
- Shift 3 units to the right.
- Shift 3 units to the left.
- Reflect about the x -axis.
- Reflect about the y -axis.
- Stretch vertically by a factor of 3.
- Shrink vertically by a factor of 3.

2. Explain how each graph is obtained from the graph of $y = f(x)$.

- | | |
|---------------------|----------------------------|
| (a) $y = f(x) + 8$ | (b) $y = f(x + 8)$ |
| (c) $y = 8f(x)$ | (d) $y = f(8x)$ |
| (e) $y = -f(x) - 1$ | (f) $y = 8f(\frac{1}{8}x)$ |

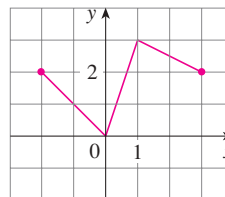
3. The graph of $y = f(x)$ is given. Match each equation with its graph and give reasons for your choices.

- | | |
|---------------------------|---------------------|
| (a) $y = f(x - 4)$ | (b) $y = f(x) + 3$ |
| (c) $y = \frac{1}{3}f(x)$ | (d) $y = -f(x + 4)$ |
| (e) $y = 2f(x + 6)$ | |



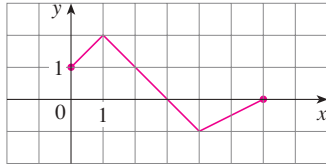
4. The graph of f is given. Draw the graphs of the following functions.

- | | |
|---------------------------|--------------------|
| (a) $y = f(x) - 3$ | (b) $y = f(x + 1)$ |
| (c) $y = \frac{1}{2}f(x)$ | (d) $y = -f(x)$ |

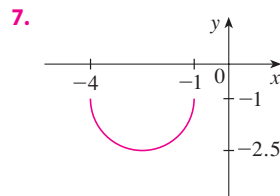
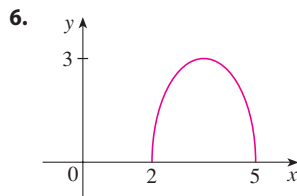
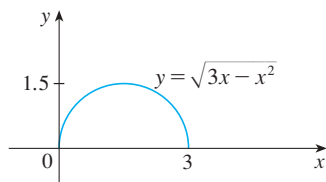


5. The graph of f is given. Use it to graph the following functions.

(a) $y = f(2x)$ (b) $y = f(\frac{1}{2}x)$
 (c) $y = f(-x)$ (d) $y = -f(-x)$



- 6–7 The graph of $y = \sqrt{3x - x^2}$ is given. Use transformations to create a function whose graph is as shown.



8. (a) How is the graph of $y = 2 \sin x$ related to the graph of $y = \sin x$? Use your answer and Figure 6 to sketch the graph of $y = 2 \sin x$.
 (b) How is the graph of $y = 1 + \sqrt{x}$ related to the graph of $y = \sqrt{x}$? Use your answer and Figure 4(a) to sketch the graph of $y = 1 + \sqrt{x}$.

9–24 Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions given in Section 1.2, and then applying the appropriate transformations.

9. $y = -x^2$ 10. $y = (x - 3)^2$
 11. $y = x^3 + 1$ 12. $y = 1 - \frac{1}{x}$
 13. $y = 2 \cos 3x$ 14. $y = 2\sqrt{x + 1}$
 15. $y = x^2 - 4x + 5$ 16. $y = 1 + \sin \pi x$

17. $y = 2 - \sqrt{x}$

18. $y = 3 - 2 \cos x$

19. $y = \sin(\frac{1}{2}x)$

20. $y = |x| - 2$

21. $y = |x - 2|$

22. $y = \frac{1}{4} \tan\left(x - \frac{\pi}{4}\right)$

23. $y = |\sqrt{x} - 1|$

24. $y = |\cos \pi x|$

25. The city of New Orleans is located at latitude 30°N . Use Figure 9 to find a function that models the number of hours of daylight at New Orleans as a function of the time of year. To check the accuracy of your model, use the fact that on March 31 the sun rises at 5:51 AM and sets at 6:18 PM in New Orleans.

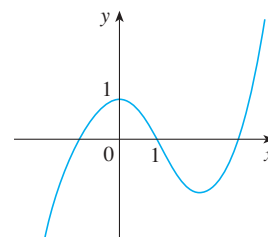
26. A variable star is one whose brightness alternately increases and decreases. For the most visible variable star, Delta Cephei, the time between periods of maximum brightness is 5.4 days, the average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude. Find a function that models the brightness of Delta Cephei as a function of time.

27. Some of the highest tides in the world occur in the Bay of Fundy on the Atlantic Coast of Canada. At Hopewell Cape the water depth at low tide is about 2.0 m and at high tide it is about 12.0 m. The natural period of oscillation is about 12 hours and on June 30, 2009, high tide occurred at 6:45 AM. Find a function involving the cosine function that models the water depth $D(t)$ (in meters) as a function of time t (in hours after midnight) on that day.

28. In a normal respiratory cycle the volume of air that moves into and out of the lungs is about 500 mL. The reserve and residue volumes of air that remain in the lungs occupy about 2000 mL and a single respiratory cycle for an average human takes about 4 seconds. Find a model for the total volume of air $V(t)$ in the lungs as a function of time.

29. (a) How is the graph of $y = f(|x|)$ related to the graph of f ?
 (b) Sketch the graph of $y = \sin |x|$.
 (c) Sketch the graph of $y = \sqrt{|x|}$.

30. Use the given graph of f to sketch the graph of $y = 1/f(x)$. Which features of f are the most important in sketching $y = 1/f(x)$? Explain how they are used.



31–32 Find (a) $f + g$, (b) $f - g$, (c) fg , and (d) f/g and state their domains.

31. $f(x) = x^3 + 2x^2, \quad g(x) = 3x^2 - 1$

32. $f(x) = \sqrt{3 - x}, \quad g(x) = \sqrt{x^2 - 1}$

33–38 Find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$ and their domains.

33. $f(x) = 3x + 5, \quad g(x) = x^2 + x$

34. $f(x) = x^3 - 2, \quad g(x) = 1 - 4x$

35. $f(x) = \sqrt{x + 1}, \quad g(x) = 4x - 3$

36. $f(x) = \sin x, \quad g(x) = x^2 + 1$

37. $f(x) = x + \frac{1}{x}, \quad g(x) = \frac{x + 1}{x + 2}$

38. $f(x) = \frac{x}{1 + x}, \quad g(x) = \sin 2x$

39–42 Find $f \circ g \circ h$.

39. $f(x) = 3x - 2, \quad g(x) = \sin x, \quad h(x) = x^2$

40. $f(x) = |x - 4|, \quad g(x) = 2^x, \quad h(x) = \sqrt{x}$

41. $f(x) = \sqrt{x - 3}, \quad g(x) = x^2, \quad h(x) = x^3 + 2$

42. $f(x) = \tan x, \quad g(x) = \frac{x}{x - 1}, \quad h(x) = \sqrt[3]{x}$

43–48 Express the function in the form $f \circ g$.

43. $F(x) = (2x + x^2)^4$

44. $F(x) = \cos^2 x$

45. $F(x) = \frac{\sqrt[3]{x}}{1 + \sqrt[3]{x}}$

46. $G(x) = \sqrt[3]{\frac{x}{1 + x}}$

47. $v(t) = \sec(t^2) \tan(t^2)$

48. $u(t) = \frac{\tan t}{1 + \tan t}$

49–51 Express the function in the form $f \circ g \circ h$.

49. $R(x) = \sqrt{\sqrt{x} - 1}$

50. $H(x) = \sqrt[3]{2 + |x|}$

51. $S(t) = \sin^2(\cos t)$

52. Use the table to evaluate each expression.

- $f(g(1))$

$g(f(1))$

$f(f(1))$
- $g(g(1))$

$(g \circ f)(3)$

$(f \circ g)(6)$

x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

53. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.

- $f(g(2))$

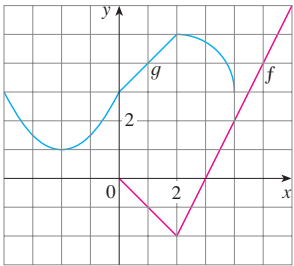
$g(f(0))$

$(f \circ g)(0)$

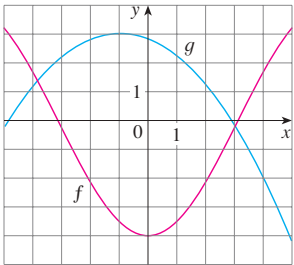
- $(g \circ f)(6)$

$(g \circ g)(-2)$

$(f \circ f)(4)$



54. Use the given graphs of f and g to estimate the value of $f(g(x))$ for $x = -5, -4, -3, \dots, 5$. Use these estimates to sketch a rough graph of $f \circ g$.



55. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.

- (a) Express the radius r of this circle as a function of the time t (in seconds).

(b) If A is the area of this circle as a function of the radius, find $A \circ r$ and interpret it.

56. A spherical balloon is being inflated and the radius of the balloon is increasing at a rate of 2 cm/s.

- (a) Express the radius r of the balloon as a function of the time t (in seconds).

(b) If V is the volume of the balloon as a function of the radius, find $V \circ r$ and interpret it.

57. A ship is moving at a speed of 30 km/h parallel to a straight shoreline. The ship is 6 km from shore and it passes a lighthouse at noon.

- (a) Express the distance s between the lighthouse and the ship as a function of d , the distance the ship has traveled since noon; that is, find f so that $s = f(d)$.

(b) Express d as a function of t , the time elapsed since noon; that is, find g so that $d = g(t)$.

(c) Find $f \circ g$. What does this function represent?

58. An airplane is flying at a speed of 350 mi/h at an altitude of one mile and passes directly over a radar station at time $t = 0$.

- (a) Express the horizontal distance d (in miles) that the plane has flown as a function of t .

(b) Express the distance s between the plane and the radar station as a function of d .

(c) Use composition to express s as a function of t .

59. The **Heaviside function** H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

- Sketch the graph of the Heaviside function.
 - Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.
 - Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)
60. The Heaviside function defined in Exercise 59 can also be used to define the **ramp function** $y = tH(t)$, which represents a gradual increase in voltage or current in a circuit.
- Sketch the graph of the ramp function $y = tH(t)$.
 - Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 60$.
- Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 32$.
 - Let f and g be linear functions with equations $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. Is $f \circ g$ also a linear function? If so, what is the slope of its graph?
 - If you invest x dollars at 4% interest compounded annually, then the amount $A(x)$ of the investment after one year is $A(x) = 1.04x$. Find $A \circ A$, $A \circ A \circ A$, and $A \circ A \circ A \circ A$. What do these compositions represent? Find a formula for the composition of n copies of A .
 - If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h .)
 - If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.
 - If $f(x) = x + 4$ and $h(x) = 4x - 1$, find a function g such that $g \circ f = h$.
 - Suppose g is an even function and let $h = f \circ g$. Is h always an even function?
 - Suppose g is an odd function and let $h = f \circ g$. Is h always an odd function? What if f is odd? What if f is even?

1.4 The Tangent and Velocity Problems

In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.

The Tangent Problem

The word *tangent* is derived from the Latin word *tangens*, which means “touching.” Thus a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact. How can this idea be made precise?

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once, as in Figure 1(a). For more complicated curves this definition is inadequate. Figure 1(b) shows two lines l and t passing through a point P on a curve C . The line l intersects C only once, but it certainly does not look like what we think of as a tangent. The line t , on the other hand, looks like a tangent but it intersects C twice.

To be specific, let's look at the problem of trying to find a tangent line t to the parabola $y = x^2$ in the following example.

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION We will be able to find an equation of the tangent line t as soon as we know its slope m . The difficulty is that we know only one point, P , on t , whereas we need two points to compute the slope. But observe that we can compute an approximation to m

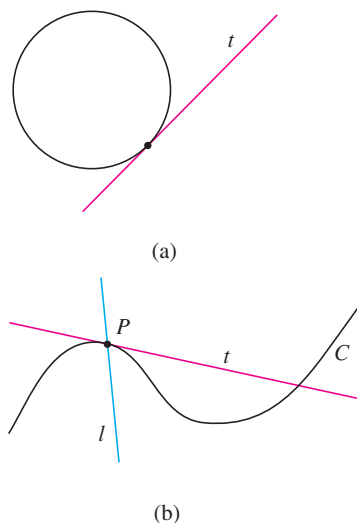


FIGURE 1

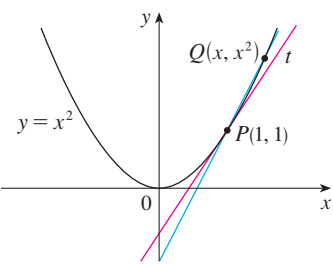


FIGURE 2

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ . [A **secant line**, from the Latin word *secans*, meaning cutting, is a line that cuts (intersects) a curve more than once.]

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

The tables in the margin show the values of m_{PQ} for several values of x close to 1. The closer Q is to P , the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2. This suggests that the slope of the tangent line t should be $m = 2$.

We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \rightarrow P} m_{PQ} = m \quad \text{and} \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line [$y - y_1 = m(x - x_1)$, see Appendix B] to write the equation of the tangent line through $(1, 1)$ as

$$y - 1 = 2(x - 1) \quad \text{or} \quad y = 2x - 1$$

Figure 3 illustrates the limiting process that occurs in this example. As Q approaches P along the parabola, the corresponding secant lines rotate about P and approach the tangent line t .

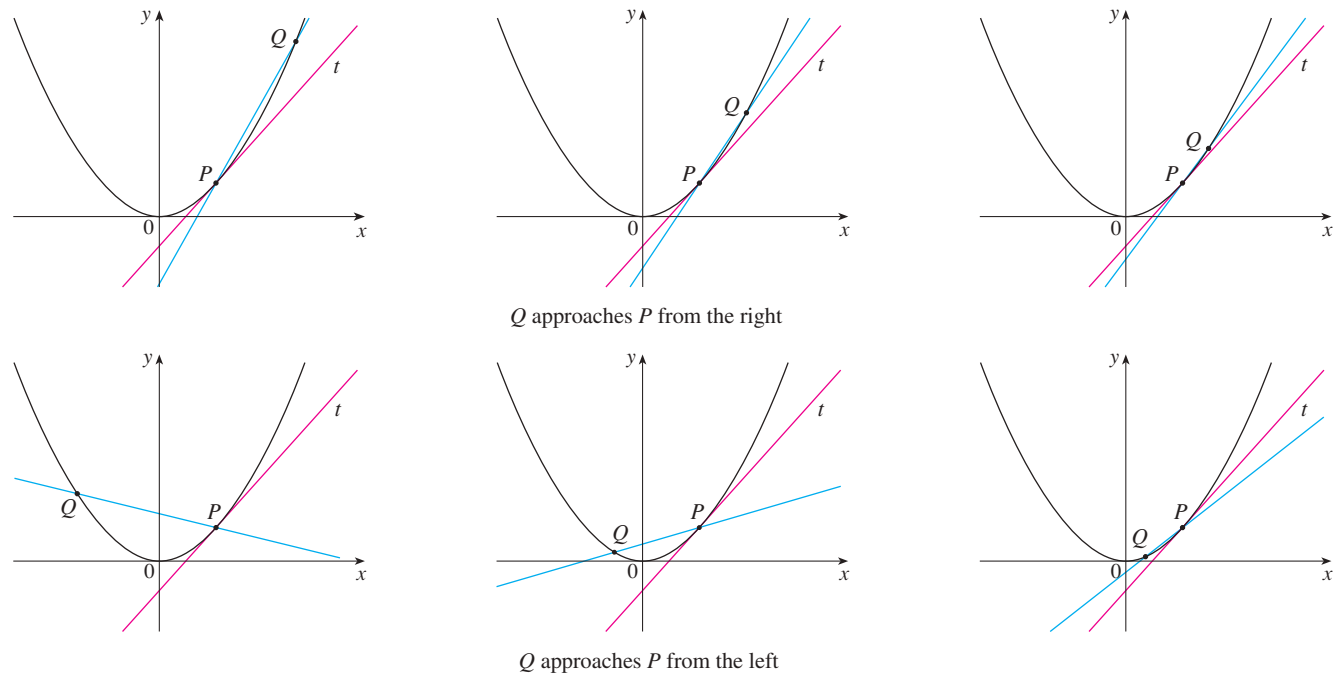


FIGURE 3

TEC In Visual 1.4 you can see how the process in Figure 3 works for additional functions.

t	Q
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

Many functions that occur in science are not described by explicit equations; they are defined by experimental data. The next example shows how to estimate the slope of the tangent line to the graph of such a function.

EXAMPLE 2 The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data in the table describe the charge Q remaining on the capacitor (measured in microcoulombs) at time t (measured in seconds after the flash goes off). Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where $t = 0.04$. [Note: The slope of the tangent line represents the electric current flowing from the capacitor to the flash bulb (measured in microamperes).]

SOLUTION In Figure 4 we plot the given data and use them to sketch a curve that approximates the graph of the function.

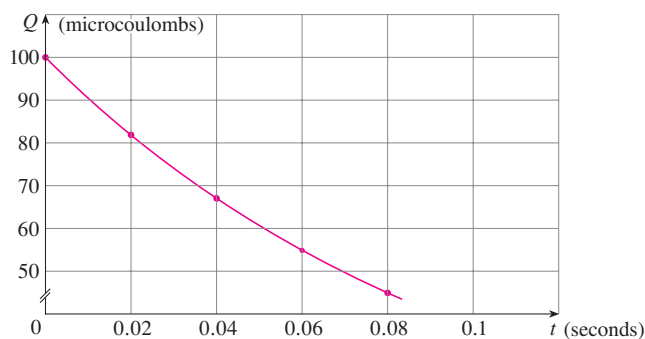


FIGURE 4

Given the points $P(0.04, 67.03)$ and $R(0.00, 100.00)$ on the graph, we find that the slope of the secant line PR is

$$m_{PR} = \frac{100.00 - 67.03}{0.00 - 0.04} = -824.25$$

R	m_{PR}
(0.00, 100.00)	-824.25
(0.02, 81.87)	-742.00
(0.06, 54.88)	-607.50
(0.08, 44.93)	-552.50
(0.10, 36.76)	-504.50

The table at the left shows the results of similar calculations for the slopes of other secant lines. From this table we would expect the slope of the tangent line at $t = 0.04$ to lie somewhere between -742 and -607.5 . In fact, the average of the slopes of the two closest secant lines is

$$\frac{1}{2}(-742 - 607.5) = -674.75$$

So, by this method, we estimate the slope of the tangent line to be about -675 .

Another method is to draw an approximation to the tangent line at P and measure the sides of the triangle ABC , as in Figure 5.

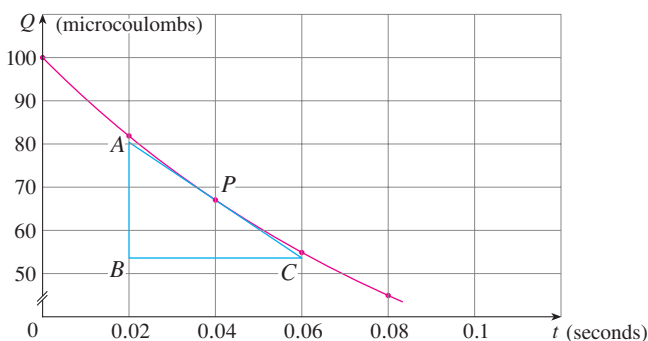


FIGURE 5

The physical meaning of the answer in Example 2 is that the electric current flowing from the capacitor to the flash bulb after 0.04 seconds is about -670 microamperes.

This gives an estimate of the slope of the tangent line as

$$-\frac{|AB|}{|BC|} \approx -\frac{80.4 - 53.6}{0.06 - 0.02} = -670$$

■ The Velocity Problem

If you watch the speedometer of a car as you travel in city traffic, you see that the speed doesn't stay the same for very long; that is, the velocity of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined? Let's investigate the example of a falling ball.

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

SOLUTION Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 seconds is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\begin{aligned} \text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}} \\ &= \frac{s(5.1) - s(5)}{0.1} \\ &= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s} \end{aligned}$$

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leq t \leq 6$	53.9
$5 \leq t \leq 5.1$	49.49
$5 \leq t \leq 5.05$	49.245
$5 \leq t \leq 5.01$	49.049
$5 \leq t \leq 5.001$	49.0049



The CN Tower in Toronto was the tallest freestanding building in the world for 32 years.

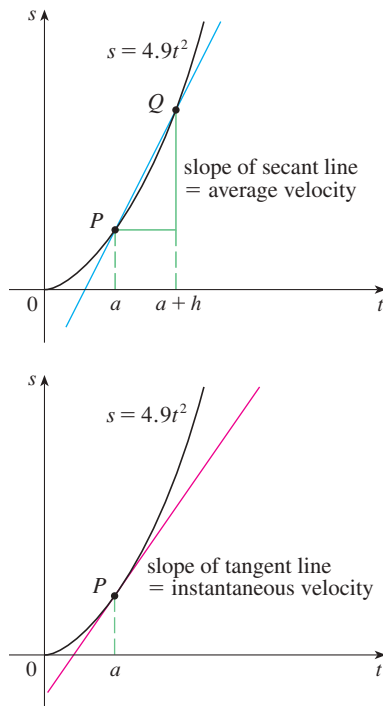


FIGURE 6

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus it appears that the (instantaneous) velocity after 5 seconds is

$$v = 49 \text{ m/s}$$

You may have the feeling that the calculations used in solving this problem are very similar to those used earlier in this section to find tangents. In fact, there is a close connection between the tangent problem and the problem of finding velocities. If we draw the graph of the distance function of the ball (as in Figure 6) and we consider the points $P(a, 4.9a^2)$ and $Q(a+h, 4.9(a+h)^2)$ on the graph, then the slope of the secant line PQ is

$$m_{PQ} = \frac{4.9(a+h)^2 - 4.9a^2}{(a+h) - a}$$

which is the same as the average velocity over the time interval $[a, a+h]$. Therefore the velocity at time $t = a$ (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).

Examples 1 and 3 show that in order to solve tangent and velocity problems we must be able to find limits. After studying methods for computing limits in the next four sections, we will return to the problems of finding tangents and velocities in Chapter 2.

1.4 EXERCISES

1. A tank holds 1000 gallons of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in gallons) after t minutes.

t (min)	5	10	15	20	25	30
V (gal)	694	444	250	111	28	0

- If P is the point $(15, 250)$ on the graph of V , find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5, 10, 20, 25$, and 30 .
 - Estimate the slope of the tangent line at P by averaging the slopes of two secant lines.
 - Use a graph of the function to estimate the slope of the tangent line at P . (This slope represents the rate at which the water is flowing from the tank after 15 minutes.)
2. A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after t min-

utes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

t (min)	36	38	40	42	44
Heartbeats	2530	2661	2806	2948	3080

The monitor estimates this value by calculating the slope of a secant line. Use the data to estimate the patient's heart rate after 42 minutes using the secant line between the points with the given values of t .

- $t = 36$ and $t = 42$
 - $t = 38$ and $t = 42$
 - $t = 40$ and $t = 42$
 - $t = 42$ and $t = 44$
- What are your conclusions?

3. The point $P(2, -1)$ lies on the curve $y = 1/(1-x)$.
- If Q is the point $(x, 1/(1-x))$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x :
 - 1.5
 - 1.9
 - 1.99
 - 1.999
 - 2.5
 - 2.1
 - 2.01
 - 2.001

- (b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, -1)$.
- (c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(2, -1)$.
4. The point $P(0.5, 0)$ lies on the curve $y = \cos \pi x$.
- (a) If Q is the point $(x, \cos \pi x)$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x :
- (i) 0 (ii) 0.4 (iii) 0.49
(iv) 0.499 (v) 1 (vi) 0.6
(vii) 0.51 (viii) 0.501
- (b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(0.5, 0)$.
- (c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(0.5, 0)$.
- (d) Sketch the curve, two of the secant lines, and the tangent line.
5. If a ball is thrown into the air with a velocity of 40 ft/s, its height in feet t seconds later is given by $y = 40t - 16t^2$.
- (a) Find the average velocity for the time period beginning when $t = 2$ and lasting
- (i) 0.5 seconds (ii) 0.1 seconds
(iii) 0.05 seconds (iv) 0.01 seconds
- (b) Estimate the instantaneous velocity when $t = 2$.
6. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters t seconds later is given by $y = 10t - 1.86t^2$.
- (a) Find the average velocity over the given time intervals:
- (i) [1, 2] (ii) [1, 1.5] (iii) [1, 1.1]
(iv) [1, 1.01] (v) [1, 1.001]
- (b) Estimate the instantaneous velocity when $t = 1$.

7. The table shows the position of a motorcyclist after accelerating from rest.

t (seconds)	0	1	2	3	4	5	6
s (feet)	0	4.9	20.6	46.5	79.2	124.8	176.7

- (a) Find the average velocity for each time period:
- (i) [2, 4] (ii) [3, 4]
(iii) [4, 5] (iv) [4, 6]
- (b) Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 3$.
8. The displacement (in centimeters) of a particle moving back and forth along a straight line is given by the equation of motion $s = 2 \sin \pi t + 3 \cos \pi t$, where t is measured in seconds.
- (a) Find the average velocity during each time period:
- (i) [1, 2] (ii) [1, 1.1]
(iii) [1, 1.01] (iv) [1, 1.001]
- (b) Estimate the instantaneous velocity of the particle when $t = 1$.
9. The point $P(1, 0)$ lies on the curve $y = \sin(10\pi/x)$.
- (a) If Q is the point $(x, \sin(10\pi/x))$, find the slope of the secant line PQ (correct to four decimal places) for $x = 2, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8$, and 0.9 . Do the slopes appear to be approaching a limit?
- (b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at P .
- (c) By choosing appropriate secant lines, estimate the slope of the tangent line at P .

1.5 The Limit of a Function

Having seen in the preceding section how limits arise when we want to find the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and numerical and graphical methods for computing them.

Let's investigate the behavior of the function f defined by $f(x) = x^2 - x + 2$ for values of x near 2. The following table gives values of $f(x)$ for values of x close to 2 but not equal to 2.

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

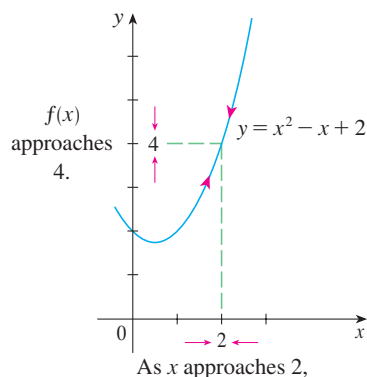


FIGURE 1

From the table and the graph of f (a parabola) shown in Figure 1 we see that the closer x is to 2 (on either side of 2), the closer $f(x)$ is to 4. In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2. We express this by saying “the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4.” The notation for this is

$$\lim_{x \rightarrow 2} (x^2 - x + 2) = 4$$

In general, we use the following notation.

1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a . (This means that f is defined on some open interval that contains a , except possibly at a itself.) Then we write

$$\lim_{x \rightarrow a} f(x) = L$$

and say “the limit of $f(x)$, as x approaches a , equals L ”

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a .

Roughly speaking, this says that the values of $f(x)$ approach L as x approaches a . In other words, the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$. (A more precise definition will be given in Section 1.7.)

An alternative notation for

$$\lim_{x \rightarrow a} f(x) = L$$

is $f(x) \rightarrow L$ as $x \rightarrow a$

which is usually read “ $f(x)$ approaches L as x approaches a .”

Notice the phrase “but $x \neq a$ ” in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a , we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near* a .

Figure 2 shows the graphs of three functions. Note that in part (c), $f(a)$ is not defined and in part (b), $f(a) \neq L$. But in each case, regardless of what happens at a , it is true that $\lim_{x \rightarrow a} f(x) = L$.

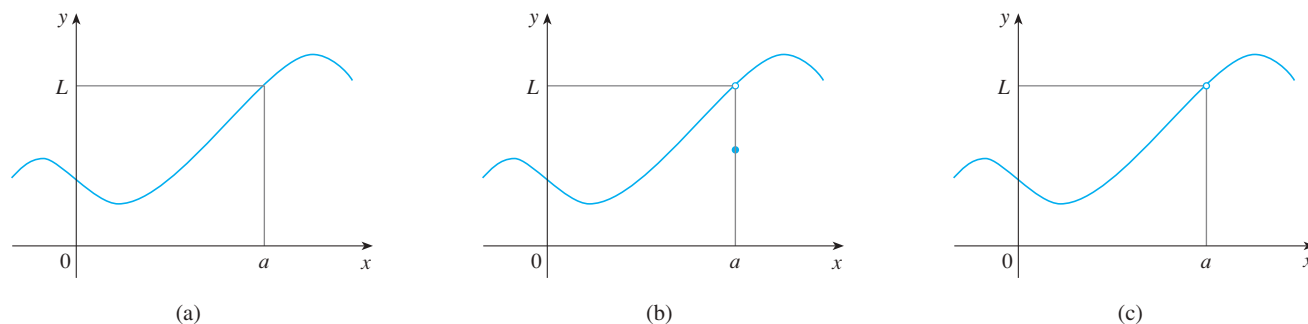


FIGURE 2 $\lim_{x \rightarrow a} f(x) = L$ in all three cases

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975



1



0.5

EXAMPLE 1 Guess the value of $\lim_{x \rightarrow 1} \frac{x - 1}{x^2 - 1}$.

SOLUTION Notice that the function $f(x) = (x - 1)/(x^2 - 1)$ is not defined when $x = 1$, but that doesn't matter because the definition of $\lim_{x \rightarrow a} f(x)$ says that we consider values of x that are close to a but not equal to a .

The tables at the left give values of $f(x)$ (correct to six decimal places) for values of x that approach 1 (but are not equal to 1). On the basis of the values in the tables, we make the guess that

$$\lim_{x \rightarrow 1} \frac{x - 1}{x^2 - 1} = 0.5$$

Example 1 is illustrated by the graph of f in Figure 3. Now let's change f slightly by giving it the value 2 when $x = 1$ and calling the resulting function g :

$$g(x) = \begin{cases} \frac{x - 1}{x^2 - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

This new function g still has the same limit as x approaches 1. (See Figure 4.)

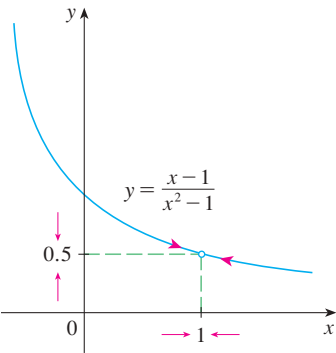


FIGURE 3

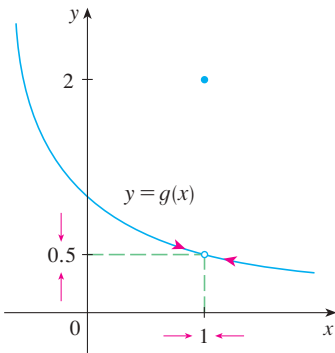


FIGURE 4

EXAMPLE 2 Estimate the value of $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

SOLUTION The table lists values of the function for several values of t near 0.

t	$\frac{\sqrt{t^2 + 9} - 3}{t^2}$
± 1.0	0.162277 ...
± 0.5	0.165525 ...
± 0.1	0.166620 ...
± 0.05	0.166655 ...
± 0.01	0.166666 ...

As t approaches 0, the values of the function seem to approach 0.166666 ... and so we guess that

$$\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} = \frac{1}{6}$$

t	$\frac{\sqrt{t^2 + 9} - 3}{t^2}$
± 0.001	0.166667
± 0.0001	0.166670
± 0.00001	0.167000
± 0.000001	0.000000

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For a further explanation of why calculators sometimes give false values, click on *Lies My Calculator and Computer Told Me*. In particular, see the section called *The Perils of Subtraction*.

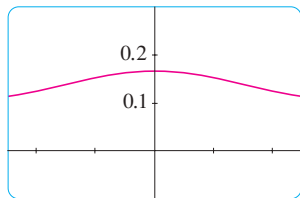
In Example 2 what would have happened if we had taken even smaller values of t ? The table in the margin shows the results from one calculator; you can see that something strange seems to be happening.

If you try these calculations on your own calculator you might get different values, but eventually you will get the value 0 if you make t sufficiently small. Does this mean that the answer is really 0 instead of $\frac{1}{6}$? No, the value of the limit is $\frac{1}{6}$, as we will show in the next section. The problem is that the **calculator gave false values** because $\sqrt{t^2 + 9}$ is very close to 3 when t is small. (In fact, when t is sufficiently small, a calculator's value for $\sqrt{t^2 + 9}$ is 3.000... to as many digits as the calculator is capable of carrying.)

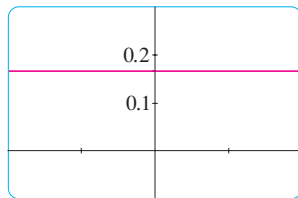
Something similar happens when we try to graph the function

$$f(t) = \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

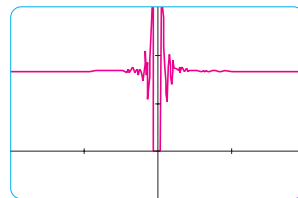
of Example 2 on a graphing calculator or computer. Parts (a) and (b) of Figure 5 show quite accurate graphs of f , and when we use the trace mode (if available) we can estimate easily that the limit is about $\frac{1}{6}$. But if we zoom in too much, as in parts (c) and (d), then we get inaccurate graphs, again because of rounding errors from the subtraction.



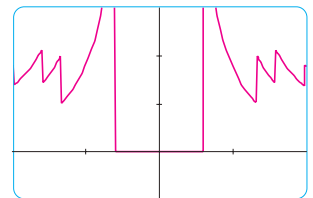
(a) $-5 \leq t \leq 5$



(b) $-0.1 \leq t \leq 0.1$



(c) $-10^{-6} \leq t \leq 10^{-6}$



(d) $-10^{-7} \leq t \leq 10^{-7}$

FIGURE 5

EXAMPLE 3 Guess the value of $\lim_{x \rightarrow 0} \frac{\sin x}{x}$.

SOLUTION The function $f(x) = (\sin x)/x$ is not defined when $x = 0$. Using a calculator (and remembering that, if $x \in \mathbb{R}$, $\sin x$ means the sine of the angle whose *radian* measure is x), we construct a table of values correct to eight decimal places. From the table at the left and the graph in Figure 6 we guess that

x	$\frac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

This guess is in fact correct, as will be proved in Chapter 2 using a geometric argument.

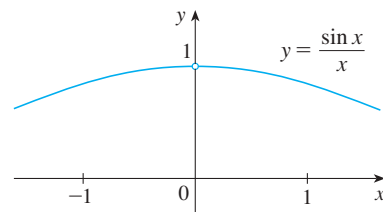


FIGURE 6

EXAMPLE 4 Investigate $\lim_{x \rightarrow 0} \sin \frac{\pi}{x}$.

Computer Algebra Systems

Computer algebra systems (CAS) have commands that compute limits. In order to avoid the types of pitfalls demonstrated in Examples 2, 4, and 5, they don't find limits by numerical experimentation. Instead, they use more sophisticated techniques such as computing infinite series. If you have access to a CAS, use the limit command to compute the limits in the examples of this section and to check your answers in the exercises of this chapter.

SOLUTION Again the function $f(x) = \sin(\pi/x)$ is undefined at 0. Evaluating the function for some small values of x , we get

$f(1) = \sin \pi = 0$

$f(\frac{1}{2}) = \sin 2\pi = 0$

$f(\frac{1}{3}) = \sin 3\pi = 0$

$f(\frac{1}{4}) = \sin 4\pi = 0$

$f(0.1) = \sin 10\pi = 0$

$f(0.01) = \sin 100\pi = 0$

Similarly, $f(0.001) = f(0.0001) = 0$. On the basis of this information we might be tempted to guess that

$$\lim_{x \rightarrow 0} \sin \frac{\pi}{x} = 0$$

but this time **our guess is wrong**. Note that although $f(1/n) = \sin n\pi = 0$ for any integer n , it is also true that $f(x) = 1$ for infinitely many values of x (such as $2/5$ or $2/101$) that approach 0. You can see this from the graph of f shown in Figure 7.

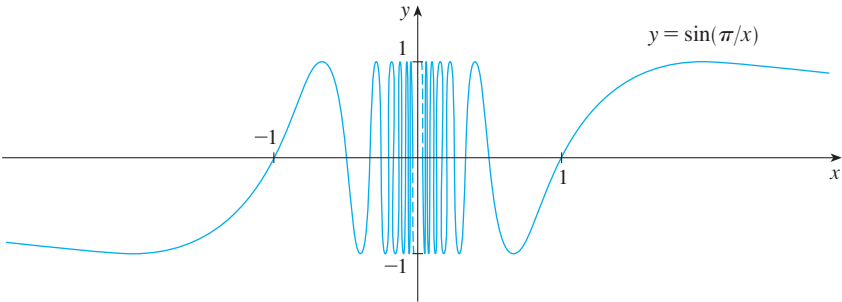


FIGURE 7

The dashed lines near the y -axis indicate that the values of $\sin(\pi/x)$ oscillate between 1 and -1 infinitely often as x approaches 0. (See Exercise 45.) Since the values of $f(x)$ do not approach a fixed number as x approaches 0,

$$\lim_{x \rightarrow 0} \sin \frac{\pi}{x} \text{ does not exist}$$

x	$x^3 + \frac{\cos 5x}{10,000}$
1	1.000028
0.5	0.124920
0.1	0.001088
0.05	0.000222
0.01	0.000101

EXAMPLE 5 Find $\lim_{x \rightarrow 0} \left(x^3 + \frac{\cos 5x}{10,000} \right)$.

SOLUTION As before, we construct a table of values. From the first table in the margin it appears that

$$\lim_{x \rightarrow 0} \left(x^3 + \frac{\cos 5x}{10,000} \right) = 0$$

But if we persevere with smaller values of x , the second table suggests that

$$\lim_{x \rightarrow 0} \left(x^3 + \frac{\cos 5x}{10,000} \right) = 0.000100 = \frac{1}{10,000}$$

x	$x^3 + \frac{\cos 5x}{10,000}$
0.005	0.0001009
0.001	0.0001000

In Section 1.8 we will see that $\lim_{x \rightarrow 0} \cos 5x = 1$; then it follows that the limit is 0.0001.

Examples 4 and 5 illustrate some of **the pitfalls in guessing the value of a limit**. It is easy to guess the wrong value if we use inappropriate values of x , but it is difficult to

know when to stop calculating values. And, as the discussion after Example 2 shows, sometimes calculators and computers give the wrong values. In the next section, however, we will develop foolproof methods for calculating limits.

One-Sided Limits

EXAMPLE 6 The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

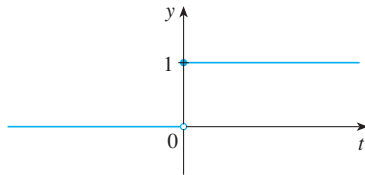


FIGURE 8
The Heaviside function

[This function is named after the electrical engineer Oliver Heaviside (1850–1925) and can be used to describe an electric current that is switched on at time $t = 0$.] Its graph is shown in Figure 8.

As t approaches 0 from the left, $H(t)$ approaches 0. As t approaches 0 from the right, $H(t)$ approaches 1. There is no single number that $H(t)$ approaches as t approaches 0. Therefore $\lim_{t \rightarrow 0} H(t)$ does not exist.

We noticed in Example 6 that $H(t)$ approaches 0 as t approaches 0 from the left and $H(t)$ approaches 1 as t approaches 0 from the right. We indicate this situation symbolically by writing

$$\lim_{t \rightarrow 0^-} H(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow 0^+} H(t) = 1$$

The notation $t \rightarrow 0^-$ indicates that we consider only values of t that are less than 0. Likewise, $t \rightarrow 0^+$ indicates that we consider only values of t that are greater than 0.

2 Definition of One-Sided Limits

We write

$$\lim_{x \rightarrow a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x less than a .

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a . Similarly, if we require that x be greater than a , we get “the **right-hand limit of $f(x)$ as x approaches a** is equal to L ” and we write

$$\lim_{x \rightarrow a^+} f(x) = L$$

Thus the notation $x \rightarrow a^+$ means that we consider only x greater than a . These definitions are illustrated in Figure 9.

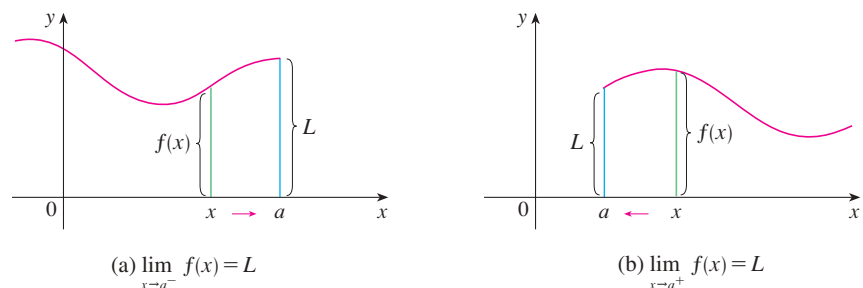


FIGURE 9

By comparing Definition 1 with the definitions of one-sided limits, we see that the following is true.

3

$\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^-} f(x) = L$ and $\lim_{x \rightarrow a^+} f(x) = L$

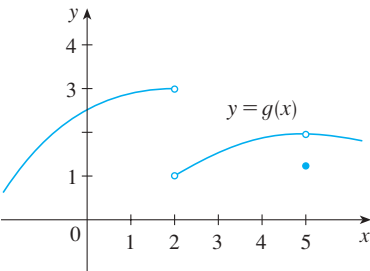


FIGURE 10

EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

- (a) $\lim_{x \rightarrow 2^-} g(x)$ (b) $\lim_{x \rightarrow 2^+} g(x)$ (c) $\lim_{x \rightarrow 2} g(x)$
- (d) $\lim_{x \rightarrow 5^-} g(x)$ (e) $\lim_{x \rightarrow 5^+} g(x)$ (f) $\lim_{x \rightarrow 5} g(x)$

SOLUTION From the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore

(a) $\lim_{x \rightarrow 2^-} g(x) = 3$ and (b) $\lim_{x \rightarrow 2^+} g(x) = 1$

(c) Since the left and right limits are different, we conclude from (3) that $\lim_{x \rightarrow 2} g(x)$ does not exist.

The graph also shows that

(d) $\lim_{x \rightarrow 5^-} g(x) = 2$ and (e) $\lim_{x \rightarrow 5^+} g(x) = 2$

(f) This time the left and right limits are the same and so, by (3), we have

$$\lim_{x \rightarrow 5} g(x) = 2$$

Despite this fact, notice that $g(5) \neq 2$. ■

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

■ Infinite Limits

EXAMPLE 8 Find $\lim_{x \rightarrow 0} \frac{1}{x^2}$ if it exists.

SOLUTION As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large. (See the table in the margin.) In fact, it appears from the graph of the function $f(x) = 1/x^2$ shown in Figure 11 that the values of $f(x)$ can be made arbitrarily large by taking x close enough to 0. Thus the values of $f(x)$ do not approach a number, so $\lim_{x \rightarrow 0} (1/x^2)$ does not exist. ■

To indicate the kind of behavior exhibited in Example 8, we use the notation

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

⚠ This does not mean that we are regarding ∞ as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist: $1/x^2$ can be made as large as we like by taking x close enough to 0.

In general, we write symbolically

$$\lim_{x \rightarrow a} f(x) = \infty$$

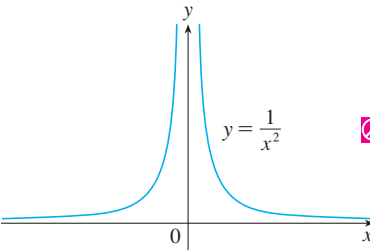


FIGURE 11

to indicate that the values of $f(x)$ tend to become larger and larger (or “increase without bound”) as x becomes closer and closer to a .

4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a , except possibly at a itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a , but not equal to a .

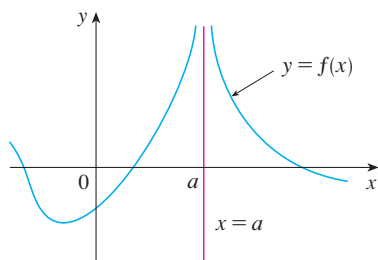


FIGURE 12

$$\lim_{x \rightarrow a} f(x) = \infty$$

When we say a number is “large negative,” we mean that it is negative but its magnitude (absolute value) is large.

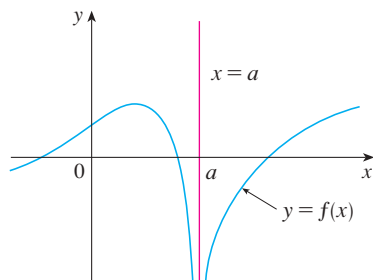


FIGURE 13

$$\lim_{x \rightarrow a} f(x) = -\infty$$

Another notation for $\lim_{x \rightarrow a} f(x) = \infty$ is

$$f(x) \rightarrow \infty \quad \text{as} \quad x \rightarrow a$$

Again, the symbol ∞ is not a number, but the expression $\lim_{x \rightarrow a} f(x) = \infty$ is often read as

“the limit of $f(x)$, as x approaches a , is infinity”

or

“ $f(x)$ becomes infinite as x approaches a ”

or

“ $f(x)$ increases without bound as x approaches a ”

This definition is illustrated graphically in Figure 12.

A similar sort of limit, for functions that become large negative as x gets close to a , is defined in Definition 5 and is illustrated in Figure 13.

5 Definition Let f be a function defined on both sides of a , except possibly at a itself. Then

$$\lim_{x \rightarrow a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a , but not equal to a .

The symbol $\lim_{x \rightarrow a} f(x) = -\infty$ can be read as “the limit of $f(x)$, as x approaches a , is negative infinity” or “ $f(x)$ decreases without bound as x approaches a .” As an example we have

$$\lim_{x \rightarrow 0} \left(-\frac{1}{x^2} \right) = -\infty$$

Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \rightarrow a^-} f(x) = \infty \qquad \lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty \qquad \lim_{x \rightarrow a^+} f(x) = -\infty$$

remembering that $x \rightarrow a^-$ means that we consider only values of x that are less than a ,

and similarly $x \rightarrow a^+$ means that we consider only $x > a$. Illustrations of these four cases are given in Figure 14.

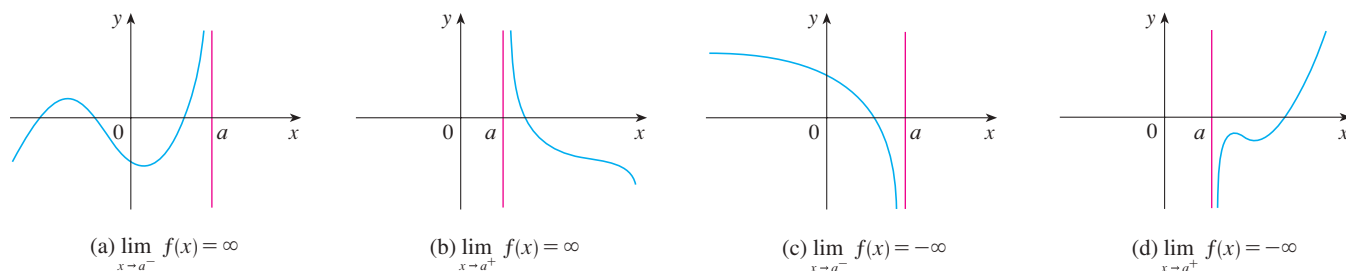


FIGURE 14

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\begin{array}{lll} \lim_{x \rightarrow a} f(x) = \infty & \lim_{x \rightarrow a^-} f(x) = \infty & \lim_{x \rightarrow a^+} f(x) = \infty \\ \lim_{x \rightarrow a} f(x) = -\infty & \lim_{x \rightarrow a^-} f(x) = -\infty & \lim_{x \rightarrow a^+} f(x) = -\infty \end{array}$$

For instance, the y -axis is a vertical asymptote of the curve $y = 1/x^2$ because $\lim_{x \rightarrow 0} (1/x^2) = \infty$. In Figure 14 the line $x = a$ is a vertical asymptote in each of the four cases shown. In general, knowledge of vertical asymptotes is very useful in sketching graphs.

EXAMPLE 9 Find $\lim_{x \rightarrow 3^+} \frac{2x}{x-3}$ and $\lim_{x \rightarrow 3^-} \frac{2x}{x-3}$.

SOLUTION If x is close to 3 but larger than 3, then the denominator $x - 3$ is a small positive number and $2x$ is close to 6. So the quotient $2x/(x - 3)$ is a large *positive* number. [For instance, if $x = 3.01$ then $2x/(x - 3) = 6.02/0.01 = 602$.] Thus, intuitively, we see that

$$\lim_{x \rightarrow 3^+} \frac{2x}{x-3} = \infty$$

Likewise, if x is close to 3 but smaller than 3, then $x - 3$ is a small negative number but $2x$ is still a positive number (close to 6). So $2x/(x - 3)$ is a numerically large *negative* number. Thus

$$\lim_{x \rightarrow 3^-} \frac{2x}{x-3} = -\infty$$

The graph of the curve $y = 2x/(x - 3)$ is given in Figure 15. The line $x = 3$ is a vertical asymptote. ■

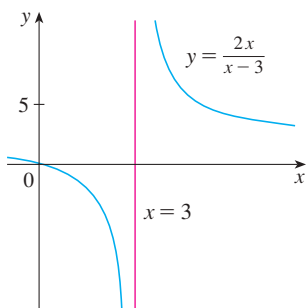


FIGURE 15

EXAMPLE 10 Find the vertical asymptotes of $f(x) = \tan x$.

SOLUTION Because

$$\tan x = \frac{\sin x}{\cos x}$$

there are potential vertical asymptotes where $\cos x = 0$. In fact, since $\cos x \rightarrow 0^+$ as $x \rightarrow (\pi/2)^-$ and $\cos x \rightarrow 0^-$ as $x \rightarrow (\pi/2)^+$, whereas $\sin x$ is positive (near 1) when x is near $\pi/2$, we have

$$\lim_{x \rightarrow (\pi/2)^-} \tan x = \infty \quad \text{and} \quad \lim_{x \rightarrow (\pi/2)^+} \tan x = -\infty$$

This shows that the line $x = \pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = \pi/2 + n\pi$, where n is an integer, are all vertical asymptotes of $f(x) = \tan x$. The graph in Figure 16 confirms this.

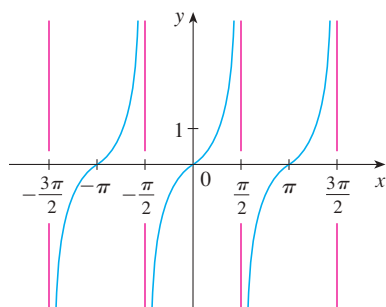


FIGURE 16

$y = \tan x$

1.5 EXERCISES

1. Explain in your own words what is meant by the equation

$$\lim_{x \rightarrow 2} f(x) = 5$$

Is it possible for this statement to be true and yet $f(2) = 3$? Explain.

2. Explain what it means to say that

$$\lim_{x \rightarrow 1^-} f(x) = 3 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = 7$$

In this situation is it possible that $\lim_{x \rightarrow 1} f(x)$ exists? Explain.

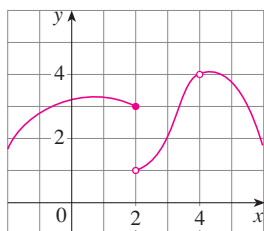
3. Explain the meaning of each of the following.

$$(a) \lim_{x \rightarrow -3} f(x) = \infty \quad (b) \lim_{x \rightarrow 4^+} f(x) = -\infty$$

4. Use the given graph of f to state the value of each quantity, if it exists. If it does not exist, explain why.

$$(a) \lim_{x \rightarrow 2^-} f(x) \quad (b) \lim_{x \rightarrow 2^+} f(x) \quad (c) \lim_{x \rightarrow 2} f(x)$$

$$(d) f(2) \quad (e) \lim_{x \rightarrow 4} f(x) \quad (f) f(4)$$

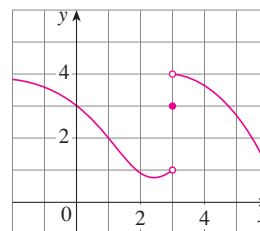


5. For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

$$(a) \lim_{x \rightarrow 1} f(x) \quad (b) \lim_{x \rightarrow 3^-} f(x) \quad (c) \lim_{x \rightarrow 3^+} f(x)$$

$$(d) \lim_{x \rightarrow 3} f(x)$$

$$(e) f(3)$$



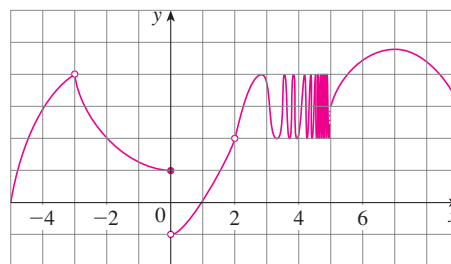
6. For the function h whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

$$(a) \lim_{x \rightarrow -3^-} h(x) \quad (b) \lim_{x \rightarrow -3^+} h(x) \quad (c) \lim_{x \rightarrow -3} h(x)$$

$$(d) h(-3) \quad (e) \lim_{x \rightarrow 0^-} h(x) \quad (f) \lim_{x \rightarrow 0^+} h(x)$$

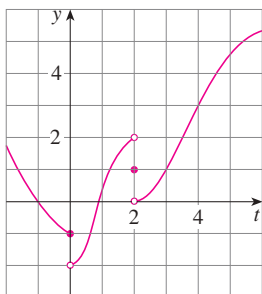
$$(g) \lim_{x \rightarrow 0} h(x) \quad (h) h(0) \quad (i) \lim_{x \rightarrow 2} h(x)$$

$$(j) h(2) \quad (k) \lim_{x \rightarrow 5^+} h(x) \quad (l) \lim_{x \rightarrow 5^-} h(x)$$



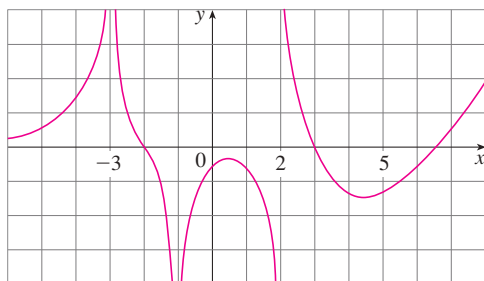
7. For the function g whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

- (a) $\lim_{t \rightarrow 0^-} g(t)$ (b) $\lim_{t \rightarrow 0^+} g(t)$ (c) $\lim_{t \rightarrow 0} g(t)$
 (d) $\lim_{t \rightarrow 2^-} g(t)$ (e) $\lim_{t \rightarrow 2^+} g(t)$ (f) $\lim_{t \rightarrow 2} g(t)$
 (g) $g(2)$ (h) $\lim_{t \rightarrow 4} g(t)$



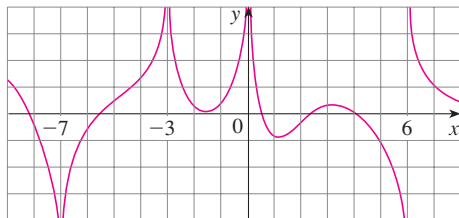
8. For the function A whose graph is shown, state the following.

- (a) $\lim_{x \rightarrow -3} A(x)$ (b) $\lim_{x \rightarrow 2^-} A(x)$
 (c) $\lim_{x \rightarrow 2^+} A(x)$ (d) $\lim_{x \rightarrow -1} A(x)$
 (e) The equations of the vertical asymptotes



9. For the function f whose graph is shown, state the following.

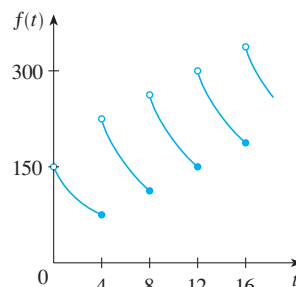
- (a) $\lim_{x \rightarrow -7} f(x)$ (b) $\lim_{x \rightarrow -3} f(x)$ (c) $\lim_{x \rightarrow 0} f(x)$
 (d) $\lim_{x \rightarrow 6^-} f(x)$ (e) $\lim_{x \rightarrow 6^+} f(x)$
 (f) The equations of the vertical asymptotes.



10. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. Find

$$\lim_{t \rightarrow 12^-} f(t) \quad \text{and} \quad \lim_{t \rightarrow 12^+} f(t)$$

and explain the significance of these one-sided limits.



- 11–12 Sketch the graph of the function and use it to determine the values of a for which $\lim_{x \rightarrow a} f(x)$ exists.

$$11. f(x) = \begin{cases} 1 + x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x < 1 \\ 2 - x & \text{if } x \geq 1 \end{cases}$$

$$12. f(x) = \begin{cases} 1 + \sin x & \text{if } x < 0 \\ \cos x & \text{if } 0 \leq x \leq \pi \\ \sin x & \text{if } x > \pi \end{cases}$$

- 13–14 Use the graph of the function f to state the value of each limit, if it exists. If it does not exist, explain why.

$$(a) \lim_{x \rightarrow 0^-} f(x) \quad (b) \lim_{x \rightarrow 0^+} f(x) \quad (c) \lim_{x \rightarrow 0} f(x)$$

$$13. f(x) = \frac{1}{1 + 2^{1/x}} \quad 14. f(x) = \frac{x^2 + x}{\sqrt{x^3 + x^2}}$$

- 15–18 Sketch the graph of an example of a function f that satisfies all of the given conditions.

$$15. \lim_{x \rightarrow 0^-} f(x) = -1, \quad \lim_{x \rightarrow 0^+} f(x) = 2, \quad f(0) = 1$$

$$16. \lim_{x \rightarrow 0} f(x) = 1, \quad \lim_{x \rightarrow 3^-} f(x) = -2, \quad \lim_{x \rightarrow 3^+} f(x) = 2, \\ f(0) = -1, \quad f(3) = 1$$

$$17. \lim_{x \rightarrow 3^+} f(x) = 4, \quad \lim_{x \rightarrow 3^-} f(x) = 2, \quad \lim_{x \rightarrow -2} f(x) = 2, \\ f(3) = 3, \quad f(-2) = 1$$

$$18. \lim_{x \rightarrow 0^-} f(x) = 2, \quad \lim_{x \rightarrow 0^+} f(x) = 0, \quad \lim_{x \rightarrow 4^-} f(x) = 3, \\ \lim_{x \rightarrow 4^+} f(x) = 0, \quad f(0) = 2, \quad f(4) = 1$$

- 19–22 Guess the value of the limit (if it exists) by evaluating the function at the given numbers (correct to six decimal places).

$$19. \lim_{x \rightarrow 3} \frac{x^2 - 3x}{x^2 - 9}, \\ x = 3.1, 3.05, 3.01, 3.001, 3.0001, \\ 2.9, 2.95, 2.99, 2.999, 2.9999$$

20. $\lim_{x \rightarrow -3} \frac{x^2 - 3x}{x^2 - 9}$,
 $x = -2.5, -2.9, -2.95, -2.99, -2.999, -2.9999,$
 $-3.5, -3.1, -3.05, -3.01, -3.001, -3.0001$
21. $\lim_{x \rightarrow 0} \frac{\sin x}{x + \tan x}$, $x = \pm 1, \pm 0.5, \pm 0.2, \pm 0.1, \pm 0.05, \pm 0.01$
22. $\lim_{h \rightarrow 0} \frac{(2 + h)^5 - 32}{h}$,
 $h = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$

23–26 Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

23. $\lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{\tan 2\theta}$
24. $\lim_{p \rightarrow -1} \frac{1 + p^9}{1 + p^{15}}$
25. $\lim_{x \rightarrow 0^+} x^x$
26. $\lim_{t \rightarrow 0} \frac{5^t - 1}{t}$

-  **27.** (a) By graphing the function $f(x) = (\cos 2x - \cos x)/x^2$ and zooming in toward the point where the graph crosses the y -axis, estimate the value of $\lim_{x \rightarrow 0} f(x)$.
 (b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

-  **28.** (a) Estimate the value of

$$\lim_{x \rightarrow 0} \frac{\sin x}{\sin \pi x}$$

by graphing the function $f(x) = (\sin x)/(\sin \pi x)$. State your answer correct to two decimal places.

- (b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

29–39 Determine the infinite limit.

29. $\lim_{x \rightarrow 5^+} \frac{x + 1}{x - 5}$
30. $\lim_{x \rightarrow 5^-} \frac{x + 1}{x - 5}$
31. $\lim_{x \rightarrow 1} \frac{2 - x}{(x - 1)^2}$
32. $\lim_{x \rightarrow 3^-} \frac{\sqrt{x}}{(x - 3)^5}$
33. $\lim_{x \rightarrow -2^+} \frac{x - 1}{x^2(x + 2)}$
34. $\lim_{x \rightarrow 0} \frac{x - 1}{x^2(x + 2)}$
35. $\lim_{x \rightarrow (\pi/2)^+} \frac{1}{x} \sec x$
36. $\lim_{x \rightarrow \pi^-} \cot x$
37. $\lim_{x \rightarrow 2\pi^-} x \csc x$
38. $\lim_{x \rightarrow 2^-} \frac{x^2 - 2x}{x^2 - 4x + 4}$
39. $\lim_{x \rightarrow 2^+} \frac{x^2 - 2x - 8}{x^2 - 5x + 6}$

- 40.** (a) Find the vertical asymptotes of the function

$$y = \frac{x^2 + 1}{3x - 2x^2}$$



- (b) Confirm your answer to part (a) by graphing the function.

- 41.** Determine $\lim_{x \rightarrow 1^-} \frac{1}{x^3 - 1}$ and $\lim_{x \rightarrow 1^+} \frac{1}{x^3 - 1}$

- (a) by evaluating $f(x) = 1/(x^3 - 1)$ for values of x that approach 1 from the left and from the right,
 (b) by reasoning as in Example 9, and
 (c) from a graph of f .



- 42.** (a) By graphing the function $f(x) = (\tan 4x)/x$ and zooming in toward the point where the graph crosses the y -axis, estimate the value of $\lim_{x \rightarrow 0} f(x)$.
 (b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

- 43.** (a) Evaluate the function $f(x) = x^2 - (2^x/1000)$ for $x = 1, 0.8, 0.6, 0.4, 0.2, 0.1$, and 0.05 , and guess the value of

$$\lim_{x \rightarrow 0} \left(x^2 - \frac{2^x}{1000} \right)$$

- (b) Evaluate $f(x)$ for $x = 0.04, 0.02, 0.01, 0.005, 0.003$, and 0.001 . Guess again.

- 44.** (a) Evaluate $h(x) = (\tan x - x)/x^3$ for $x = 1, 0.5, 0.1, 0.05, 0.01$, and 0.005 .

- (b) Guess the value of $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$.

- (c) Evaluate $h(x)$ for successively smaller values of x until you finally reach a value of 0 for $h(x)$. Are you still confident that your guess in part (b) is correct? Explain why you eventually obtained values of 0 for $h(x)$. (In Section 6.8 a method for evaluating this limit will be explained.)



- (d) Graph the function h in the viewing rectangle $[-1, 1]$ by $[0, 1]$. Then zoom in toward the point where the graph crosses the y -axis to estimate the limit of $h(x)$ as x approaches 0. Continue to zoom in until you observe distortions in the graph of h . Compare with the results of part (c).



- 45.** Graph the function $f(x) = \sin(\pi/x)$ of Example 4 in the viewing rectangle $[-1, 1]$ by $[-1, 1]$. Then zoom in toward the origin several times. Comment on the behavior of this function.

- 46.** Consider the function $f(x) = \tan \frac{1}{x}$.

- (a) Show that $f(x) = 0$ for $x = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \dots$

- (b) Show that $f(x) = 1$ for $x = \frac{4}{\pi}, \frac{4}{5\pi}, \frac{4}{9\pi}, \dots$

- (c) What can you conclude about $\lim_{x \rightarrow 0^+} \tan \frac{1}{x}$?

-  **47.** Use a graph to estimate the equations of all the vertical asymptotes of the curve

$$y = \tan(2 \sin x) \quad -\pi \leq x \leq \pi$$

Then find the exact equations of these asymptotes.

- 48.** In the theory of relativity, the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \rightarrow c^-$?

-  **49.** (a) Use numerical and graphical evidence to guess the value of the limit

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{\sqrt{x} - 1}$$

- (b) How close to 1 does x have to be to ensure that the function in part (a) is within a distance 0.5 of its limit?

1.6 Calculating Limits Using the Limit Laws

In Section 1.5 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answer. In this section we use the following properties of limits, called the *Limit Laws*, to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \rightarrow a} f(x) \quad \text{and} \quad \lim_{x \rightarrow a} g(x)$$

exist. Then

1. $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2. $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3. $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$
4. $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
5. $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ if $\lim_{x \rightarrow a} g(x) \neq 0$

These five laws can be stated verbally as follows:

- | | |
|---|--|
| <p>Sum Law</p> <p>Difference Law</p> <p>Constant Multiple Law</p> <p>Product Law</p> <p>Quotient Law</p> | <ol style="list-style-type: none"> 1. The limit of a sum is the sum of the limits. 2. The limit of a difference is the difference of the limits. 3. The limit of a constant times a function is the constant times the limit of the function. 4. The limit of a product is the product of the limits. 5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0). |
|---|--|

It is easy to believe that these properties are true. For instance, if $f(x)$ is close to L and $g(x)$ is close to M , it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$. This gives us an intuitive basis for believing that Law 1 is true. In Section 1.7 we give a

precise definition of a limit and use it to prove this law. The proofs of the remaining laws are given in Appendix F.

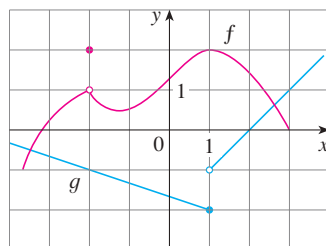


FIGURE 1

EXAMPLE 1 Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

$$(a) \lim_{x \rightarrow -2} [f(x) + 5g(x)] \quad (b) \lim_{x \rightarrow 1} [f(x)g(x)] \quad (c) \lim_{x \rightarrow 2} \frac{f(x)}{g(x)}$$

SOLUTION

(a) From the graphs of f and g we see that

$$\lim_{x \rightarrow -2} f(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow -2} g(x) = -1$$

Therefore we have

$$\begin{aligned} \lim_{x \rightarrow -2} [f(x) + 5g(x)] &= \lim_{x \rightarrow -2} f(x) + \lim_{x \rightarrow -2} [5g(x)] && \text{(by Limit Law 1)} \\ &= \lim_{x \rightarrow -2} f(x) + 5 \lim_{x \rightarrow -2} g(x) && \text{(by Limit Law 3)} \\ &= 1 + 5(-1) = -4 \end{aligned}$$

(b) We see that $\lim_{x \rightarrow 1} f(x) = 2$. But $\lim_{x \rightarrow 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x \rightarrow 1^-} g(x) = -2 \quad \lim_{x \rightarrow 1^+} g(x) = -1$$

So we can't use Law 4 for the desired limit. But we *can* use Law 4 for the one-sided limits:

$$\begin{aligned} \lim_{x \rightarrow 1^-} [f(x)g(x)] &= \lim_{x \rightarrow 1^-} f(x) \cdot \lim_{x \rightarrow 1^-} g(x) = 2 \cdot (-2) = -4 \\ \lim_{x \rightarrow 1^+} [f(x)g(x)] &= \lim_{x \rightarrow 1^+} f(x) \cdot \lim_{x \rightarrow 1^+} g(x) = 2 \cdot (-1) = -2 \end{aligned}$$

The left and right limits aren't equal, so $\lim_{x \rightarrow 1} [f(x)g(x)]$ does not exist.

(c) The graphs show that

$$\lim_{x \rightarrow 2} f(x) \approx 1.4 \quad \text{and} \quad \lim_{x \rightarrow 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5. The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number. ■

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law

$$6. \lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n \quad \text{where } n \text{ is a positive integer}$$

In applying these six limit laws, we need to use two special limits:

$$7. \lim_{x \rightarrow a} c = c$$

$$8. \lim_{x \rightarrow a} x = a$$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$), but proofs based on the precise definition are requested in the exercises for Section 1.7.

If we now put $f(x) = x$ in Law 6 and use Law 8, we get another useful special limit.

$$9. \lim_{x \rightarrow a} x^n = a^n \quad \text{where } n \text{ is a positive integer}$$

A similar limit holds for roots as follows. (For square roots the proof is outlined in Exercise 1.7.37.)

$$10. \lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a} \quad \text{where } n \text{ is a positive integer}$$

(If n is even, we assume that $a > 0$.)

More generally, we have the following law, which is proved in Section 1.8 as a consequence of Law 10.

Root Law

$$11. \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \quad \text{where } n \text{ is a positive integer}$$

[If n is even, we assume that $\lim_{x \rightarrow a} f(x) > 0$.]

EXAMPLE 2 Evaluate the following limits and justify each step.

$$(a) \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \qquad (b) \lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

SOLUTION

$$\begin{aligned} (a) \quad \lim_{x \rightarrow 5} (2x^2 - 3x + 4) &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 && \text{(by Laws 2 and 1)} \\ &= 2 \lim_{x \rightarrow 5} x^2 - 3 \lim_{x \rightarrow 5} x + \lim_{x \rightarrow 5} 4 && \text{(by 3)} \\ &= 2(5^2) - 3(5) + 4 && \text{(by 9, 8, and 7)} \\ &= 39 \end{aligned}$$

(b) We start by using Law 5, but its use is fully justified only at the final stage when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

$$\begin{aligned} \lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} &= \frac{\lim_{x \rightarrow -2} (x^3 + 2x^2 - 1)}{\lim_{x \rightarrow -2} (5 - 3x)} && \text{(by Law 5)} \\ &= \frac{\lim_{x \rightarrow -2} x^3 + 2 \lim_{x \rightarrow -2} x^2 - \lim_{x \rightarrow -2} 1}{\lim_{x \rightarrow -2} 5 - 3 \lim_{x \rightarrow -2} x} && \text{(by 1, 2, and 3)} \\ &= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} && \text{(by 9, 8, and 7)} \\ &= -\frac{1}{11} \end{aligned}$$

Newton and Limits

Isaac Newton was born on Christmas Day in 1642, the year of Galileo's death. When he entered Cambridge University in 1661 Newton didn't know much mathematics, but he learned quickly by reading Euclid and Descartes and by attending the lectures of Isaac Barrow. Cambridge was closed because of the plague in 1665 and 1666, and Newton returned home to reflect on what he had learned. Those two years were amazingly productive for at that time he made four of his major discoveries: (1) his representation of functions as sums of infinite series, including the binomial theorem; (2) his work on differential and integral calculus; (3) his laws of motion and law of universal gravitation; and (4) his prism experiments on the nature of light and color. Because of a fear of controversy and criticism, he was reluctant to publish his discoveries and it wasn't until 1687, at the urging of the astronomer Halley, that Newton published *Principia Mathematica*. In this work, the greatest scientific treatise ever written, Newton set forth his version of calculus and used it to investigate mechanics, fluid dynamics, and wave motion, and to explain the motion of planets and comets.

The beginnings of calculus are found in the calculations of areas and volumes by ancient Greek scholars such as Eudoxus and Archimedes. Although aspects of the idea of a limit are implicit in their "method of exhaustion," Eudoxus and Archimedes never explicitly formulated the concept of a limit. Likewise, mathematicians such as Cavalieri, Fermat, and Barrow, the immediate precursors of Newton in the development of calculus, did not actually use limits. It was Isaac Newton who was the first to talk explicitly about limits. He explained that the main idea behind limits is that quantities "approach nearer than by any given difference." Newton stated that the limit was the basic concept in calculus, but it was left to later mathematicians like Cauchy to clarify his ideas about limits.

NOTE If we let $f(x) = 2x^2 - 3x + 4$, then $f(5) = 39$. In other words, we would have gotten the correct answer in Example 2(a) by substituting 5 for x . Similarly, direct substitution provides the correct answer in part (b). The functions in Example 2 are a polynomial and a rational function, respectively, and similar use of the Limit Laws proves that direct substitution always works for such functions (see Exercises 57 and 58). We state this fact as follows.

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Functions with the Direct Substitution Property are called *continuous at a* and will be studied in Section 1.8. However, not all limits can be evaluated by direct substitution, as the following examples show.

EXAMPLE 3 Find $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$.

SOLUTION Let $f(x) = (x^2 - 1)/(x - 1)$. We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined. Nor can we apply the Quotient Law, because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$

The numerator and denominator have a common factor of $x - 1$. When we take the limit as x approaches 1, we have $x \neq 1$ and so $x - 1 \neq 0$. Therefore we can cancel the common factor and then compute the limit by direct substitution as follows:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} (x + 1) \\ &= 1 + 1 = 2 \end{aligned}$$

The limit in this example arose in Example 1.4.1 when we were trying to find the tangent to the parabola $y = x^2$ at the point $(1, 1)$. ■

NOTE 1 In Example 3 we do not have an infinite limit even though the denominator approaches 0 as $x \rightarrow 1$. When both numerator and denominator approach 0, the limit may be infinite or it may be some finite value.

NOTE 2 In Example 3 we were able to compute the limit by replacing the given function $f(x) = (x^2 - 1)/(x - 1)$ by a simpler function, $g(x) = x + 1$, with the same limit. This is valid because $f(x) = g(x)$ except when $x = 1$, and in computing a limit as x approaches 1 we don't consider what happens when x is actually *equal* to 1. In general, we have the following useful fact.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$, provided the limits exist.

EXAMPLE 4 Find $\lim_{x \rightarrow 1} g(x)$ where

$$g(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$$

SOLUTION Here g is defined at $x = 1$ and $g(1) = \pi$, but the value of a limit as x approaches 1 does not depend on the value of the function at 1. Since $g(x) = x + 1$ for $x \neq 1$, we have

$$\lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} (x + 1) = 2$$

Note that the values of the functions in Examples 3 and 4 are identical except when $x = 1$ (see Figure 2) and so they have the same limit as x approaches 1.

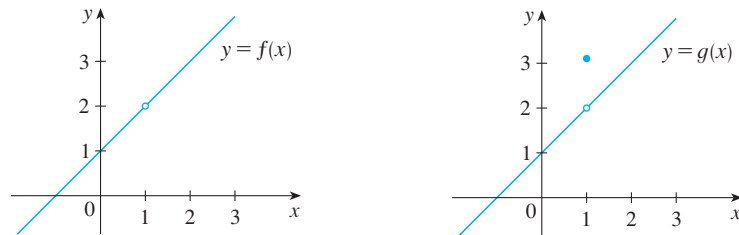


FIGURE 2

The graphs of the functions f (from Example 3) and g (from Example 4)

EXAMPLE 5 Evaluate $\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h}$.

SOLUTION If we define

$$F(h) = \frac{(3 + h)^2 - 9}{h}$$

then, as in Example 3, we can't compute $\lim_{h \rightarrow 0} F(h)$ by letting $h = 0$ since $F(0)$ is undefined. But if we simplify $F(h)$ algebraically, we find that

$$F(h) = \frac{(9 + 6h + h^2) - 9}{h} = \frac{6h + h^2}{h} = \frac{h(6 + h)}{h} = 6 + h$$

(Recall that we consider only $h \neq 0$ when letting h approach 0.) Thus

$$\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h} = \lim_{h \rightarrow 0} (6 + h) = 6$$

EXAMPLE 6 Find $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

SOLUTION We can't apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned}
 \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} &= \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3} \\
 &= \lim_{t \rightarrow 0} \frac{(t^2 + 9) - 9}{t^2(\sqrt{t^2 + 9} + 3)} \\
 &= \lim_{t \rightarrow 0} \frac{t^2}{t^2(\sqrt{t^2 + 9} + 3)} \\
 &= \lim_{t \rightarrow 0} \frac{1}{\sqrt{t^2 + 9} + 3} \\
 &= \frac{1}{\sqrt{\lim_{t \rightarrow 0} (t^2 + 9)} + 3} \\
 &= \frac{1}{3 + 3} = \frac{1}{6}
 \end{aligned}$$

Here we use several properties of limits (5, 1, 10, 7, 9).

This calculation confirms the guess that we made in Example 1.5.2. ■

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 1.5. It says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

EXAMPLE 7 Show that $\lim_{x \rightarrow 0} |x| = 0$.

SOLUTION Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The result of Example 7 looks plausible from Figure 3.

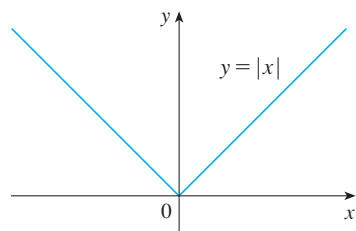


FIGURE 3

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ and so

$$\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) = 0$$

Therefore, by Theorem 1,

$$\lim_{x \rightarrow 0} |x| = 0 \quad \text{span style="color: red;">■$$

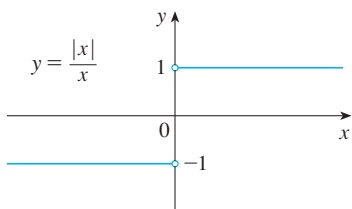


FIGURE 4

EXAMPLE 8 Prove that $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.

SOLUTION Using the facts that $|x| = x$ when $x > 0$ and $|x| = -x$ when $x < 0$, we have

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{|x|}{x} &= \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1 \\ \lim_{x \rightarrow 0^-} \frac{|x|}{x} &= \lim_{x \rightarrow 0^-} \frac{-x}{x} = \lim_{x \rightarrow 0^-} (-1) = -1\end{aligned}$$

Since the right- and left-hand limits are different, it follows from Theorem 1 that $\lim_{x \rightarrow 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ is shown in Figure 4 and supports the one-sided limits that we found. ■

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8-2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x \rightarrow 4} f(x)$ exists.

SOLUTION Since $f(x) = \sqrt{x-4}$ for $x > 4$, we have

$$\lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^-} (8 - 2x) = 8 - 2 \cdot 4 = 0$$

The right- and left-hand limits are equal. Thus the limit exists and

$$\lim_{x \rightarrow 4} f(x) = 0$$

The graph of f is shown in Figure 5. ■

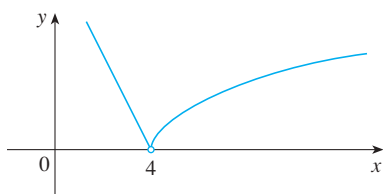


FIGURE 5

Other notations for $\lfloor x \rfloor$ are $[x]$ and $\lfloor x \rfloor$. The greatest integer function is sometimes called the *floor function*.

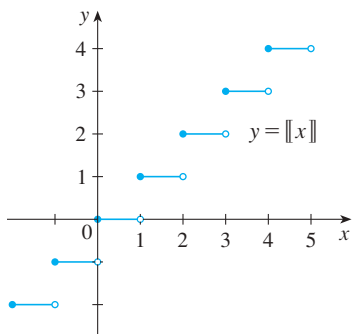


FIGURE 6
Greatest integer function

EXAMPLE 10 The **greatest integer function** is defined by $\lfloor x \rfloor$ = the largest integer that is less than or equal to x . (For instance, $\lfloor 4 \rfloor = 4$, $\lfloor 4.8 \rfloor = 4$, $\lfloor \pi \rfloor = 3$, $\lfloor \sqrt{2} \rfloor = 1$, $\lfloor -\frac{1}{2} \rfloor = -1$.) Show that $\lim_{x \rightarrow 3} \lfloor x \rfloor$ does not exist.

SOLUTION The graph of the greatest integer function is shown in Figure 6. Since $\lfloor x \rfloor = 3$ for $3 \leq x < 4$, we have

$$\lim_{x \rightarrow 3^+} \lfloor x \rfloor = \lim_{x \rightarrow 3^+} 3 = 3$$

Since $\lfloor x \rfloor = 2$ for $2 \leq x < 3$, we have

$$\lim_{x \rightarrow 3^-} \lfloor x \rfloor = \lim_{x \rightarrow 3^-} 2 = 2$$

Because these one-sided limits are not equal, $\lim_{x \rightarrow 3} \lfloor x \rfloor$ does not exist by Theorem 1. ■

The next two theorems give two additional properties of limits. Their proofs can be found in Appendix F.

2 Theorem If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a , then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$$

3 The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L$$

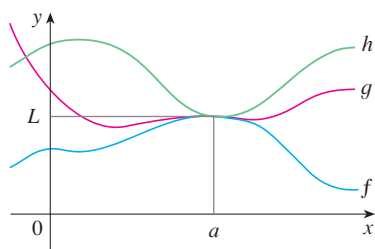


FIGURE 7

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7. It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a , and if f and h have the same limit L at a , then g is forced to have the same limit L at a .

EXAMPLE 11 Show that $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$.

SOLUTION First note that we **cannot** use



$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = \lim_{x \rightarrow 0} x^2 \cdot \lim_{x \rightarrow 0} \sin \frac{1}{x}$$

because $\lim_{x \rightarrow 0} \sin(1/x)$ does not exist (see Example 1.5.4).

Instead we apply the Squeeze Theorem, and so we need to find a function f smaller than $g(x) = x^2 \sin(1/x)$ and a function h bigger than g such that both $f(x)$ and $h(x)$ approach 0. To do this we use our knowledge of the sine function. Because the sine of any number lies between -1 and 1 , we can write.

$$\boxed{4} \quad -1 \leq \sin \frac{1}{x} \leq 1$$

Any inequality remains true when multiplied by a positive number. We know that $x^2 \geq 0$ for all x and so, multiplying each side of the inequalities in (4) by x^2 , we get

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$

as illustrated by Figure 8. We know that

$$\lim_{x \rightarrow 0} x^2 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$$

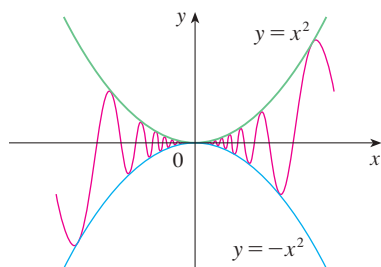


FIGURE 8

$$y = x^2 \sin(1/x)$$

1.6 EXERCISES

1. Given that

$$\lim_{x \rightarrow 2} f(x) = 4 \quad \lim_{x \rightarrow 2} g(x) = -2 \quad \lim_{x \rightarrow 2} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.

$$(a) \lim_{x \rightarrow 2} [f(x) + 5g(x)] \quad (b) \lim_{x \rightarrow 2} [g(x)]^3$$

$$(c) \lim_{x \rightarrow 2} \sqrt{f(x)} \quad (d) \lim_{x \rightarrow 2} \frac{3f(x)}{g(x)}$$

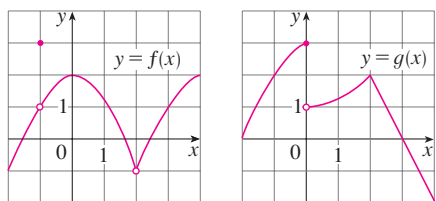
$$(e) \lim_{x \rightarrow 2} \frac{g(x)}{h(x)} \quad (f) \lim_{x \rightarrow 2} \frac{g(x)h(x)}{f(x)}$$

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

$$(a) \lim_{x \rightarrow 2} [f(x) + g(x)] \quad (b) \lim_{x \rightarrow 0} [f(x) - g(x)]$$

$$(c) \lim_{x \rightarrow -1} [f(x)g(x)] \quad (d) \lim_{x \rightarrow 3} \frac{f(x)}{g(x)}$$

$$(e) \lim_{x \rightarrow 2} [x^2 f(x)] \quad (f) f(-1) + \lim_{x \rightarrow -1} g(x)$$



3–9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

$$3. \lim_{x \rightarrow 3} (5x^3 - 3x^2 + x - 6)$$

$$4. \lim_{x \rightarrow -1} (x^4 - 3x)(x^2 + 5x + 3)$$

$$5. \lim_{t \rightarrow -2} \frac{t^4 - 2}{2t^2 - 3t + 2}$$

$$6. \lim_{u \rightarrow -2} \sqrt{u^4 + 3u + 6}$$

$$7. \lim_{x \rightarrow 8} (1 + \sqrt[3]{x})(2 - 6x^2 + x^3) \quad 8. \lim_{t \rightarrow 2} \left(\frac{t^2 - 2}{t^3 - 3t + 5} \right)^2$$

$$9. \lim_{x \rightarrow 2} \sqrt{\frac{2x^2 + 1}{3x - 2}}$$

10. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \rightarrow 2} (x + 3)$$

is correct.

11–32 Evaluate the limit, if it exists.

$$11. \lim_{x \rightarrow 5} \frac{x^2 - 6x + 5}{x - 5}$$

$$12. \lim_{x \rightarrow -3} \frac{x^2 + 3x}{x^2 - x - 12}$$

$$13. \lim_{x \rightarrow 5} \frac{x^2 - 5x + 6}{x - 5}$$

$$14. \lim_{x \rightarrow 4} \frac{x^2 + 3x}{x^2 - x - 12}$$

$$15. \lim_{t \rightarrow -3} \frac{t^2 - 9}{2t^2 + 7t + 3}$$

$$16. \lim_{x \rightarrow -1} \frac{2x^2 + 3x + 1}{x^2 - 2x - 3}$$

$$17. \lim_{h \rightarrow 0} \frac{(-5 + h)^2 - 25}{h}$$

$$18. \lim_{h \rightarrow 0} \frac{(2 + h)^3 - 8}{h}$$

$$19. \lim_{x \rightarrow -2} \frac{x + 2}{x^3 + 8}$$

$$20. \lim_{t \rightarrow 1} \frac{t^4 - 1}{t^3 - 1}$$

$$21. \lim_{h \rightarrow 0} \frac{\sqrt{9 + h} - 3}{h}$$

$$22. \lim_{u \rightarrow 2} \frac{\sqrt{4u + 1} - 3}{u - 2}$$

$$23. \lim_{x \rightarrow 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$$

$$24. \lim_{h \rightarrow 0} \frac{(3 + h)^{-1} - 3^{-1}}{h}$$

$$25. \lim_{t \rightarrow 0} \frac{\sqrt{1 + t} - \sqrt{1 - t}}{t}$$

$$26. \lim_{t \rightarrow 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right)$$

$$27. \lim_{x \rightarrow 16} \frac{4 - \sqrt{x}}{16x - x^2}$$

$$28. \lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{x^4 - 3x^2 - 4}$$

$$29. \lim_{t \rightarrow 0} \left(\frac{1}{t\sqrt{1 + t}} - \frac{1}{t} \right)$$

$$30. \lim_{x \rightarrow -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4}$$

$$31. \lim_{h \rightarrow 0} \frac{(x + h)^3 - x^3}{h}$$

$$32. \lim_{h \rightarrow 0} \frac{\frac{1}{(x + h)^2} - \frac{1}{x^2}}{h}$$

33. (a) Estimate the value of

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/(\sqrt{1 + 3x} - 1)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

34. (a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

to estimate the value of $\lim_{x \rightarrow 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

-  **35.** Use the Squeeze Theorem to show that $\lim_{x \rightarrow 0} (x^2 \cos 20\pi x) = 0$. Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$ on the same screen.

-  **36.** Use the Squeeze Theorem to show that

$$\lim_{x \rightarrow 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x} = 0$$

Illustrate by graphing the functions f , g , and h (in the notation of the Squeeze Theorem) on the same screen.

- 37.** If $4x - 9 \leq f(x) \leq x^2 - 4x + 7$ for $x \geq 0$, find $\lim_{x \rightarrow 4} f(x)$.

- 38.** If $2x \leq g(x) \leq x^4 - x^2 + 2$ for all x , evaluate $\lim_{x \rightarrow 1} g(x)$.

- 39.** Prove that $\lim_{x \rightarrow 0} x^4 \cos \frac{2}{x} = 0$.

- 40.** Prove that $\lim_{x \rightarrow 0^+} \sqrt{x} [1 + \sin^2(2\pi/x)] = 0$.

41–46 Find the limit, if it exists. If the limit does not exist, explain why.

41. $\lim_{x \rightarrow 3} (2x + |x - 3|)$

42. $\lim_{x \rightarrow -6} \frac{2x + 12}{|x + 6|}$

43. $\lim_{x \rightarrow 0.5^-} \frac{2x - 1}{|2x^3 - x^2|}$

44. $\lim_{x \rightarrow -2} \frac{2 - |x|}{2 + x}$

45. $\lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

46. $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

- 47.** The *signum* (or *sign*) *function*, denoted by sgn , is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

- (a) Sketch the graph of this function.
(b) Find each of the following limits or explain why it does not exist.

(i) $\lim_{x \rightarrow 0^+} \operatorname{sgn} x$ (ii) $\lim_{x \rightarrow 0^-} \operatorname{sgn} x$

(iii) $\lim_{x \rightarrow 0} \operatorname{sgn} x$ (iv) $\lim_{x \rightarrow 0} |\operatorname{sgn} x|$

- 48.** Let $g(x) = \operatorname{sgn}(\sin x)$.

- (a) Find each of the following limits or explain why it does not exist.

(i) $\lim_{x \rightarrow 0^+} g(x)$ (ii) $\lim_{x \rightarrow 0^-} g(x)$ (iii) $\lim_{x \rightarrow 0} g(x)$

(iv) $\lim_{x \rightarrow \pi^+} g(x)$ (v) $\lim_{x \rightarrow \pi^-} g(x)$ (vi) $\lim_{x \rightarrow \pi} g(x)$

- (b) For which values of a does $\lim_{x \rightarrow a} g(x)$ not exist?
(c) Sketch a graph of g .

49. Let $g(x) = \frac{x^2 + x - 6}{|x - 2|}$.

- (a) Find

(i) $\lim_{x \rightarrow 2^+} g(x)$ (ii) $\lim_{x \rightarrow 2^-} g(x)$

- (b) Does $\lim_{x \rightarrow 2} g(x)$ exist?

- (c) Sketch the graph of g .

- 50.** Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x - 2)^2 & \text{if } x \geq 1 \end{cases}$$

- (a) Find $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$.

- (b) Does $\lim_{x \rightarrow 1} f(x)$ exist?

- (c) Sketch the graph of f .

- 51.** Let

$$B(t) = \begin{cases} 4 - \frac{1}{2}t & \text{if } t < 2 \\ \sqrt{t + c} & \text{if } t \geq 2 \end{cases}$$

Find the value of c so that $\lim_{t \rightarrow 2} B(t)$ exists.

- 52.** Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

- (a) Evaluate each of the following, if it exists.

(i) $\lim_{x \rightarrow 1^-} g(x)$ (ii) $\lim_{x \rightarrow 1} g(x)$ (iii) $g(1)$

(iv) $\lim_{x \rightarrow 2^-} g(x)$ (v) $\lim_{x \rightarrow 2^+} g(x)$ (vi) $\lim_{x \rightarrow 2} g(x)$

- (b) Sketch the graph of g .

- 53.** (a) If the symbol $\llbracket x \rrbracket$ denotes the greatest integer function defined in Example 10, evaluate

(i) $\lim_{x \rightarrow -2^+} \llbracket x \rrbracket$ (ii) $\lim_{x \rightarrow -2} \llbracket x \rrbracket$ (iii) $\lim_{x \rightarrow -2.4} \llbracket x \rrbracket$

- (b) If n is an integer, evaluate

(i) $\lim_{x \rightarrow n^-} \llbracket x \rrbracket$ (ii) $\lim_{x \rightarrow n^+} \llbracket x \rrbracket$

- (c) For what values of a does $\lim_{x \rightarrow a} \llbracket x \rrbracket$ exist?

- 54.** Let $f(x) = \llbracket \cos x \rrbracket$, $-\pi \leq x \leq \pi$.

- (a) Sketch the graph of f .

- (b) Evaluate each limit, if it exists.

(i) $\lim_{x \rightarrow 0} f(x)$ (ii) $\lim_{x \rightarrow (\pi/2)^-} f(x)$

(iii) $\lim_{x \rightarrow (\pi/2)^+} f(x)$ (iv) $\lim_{x \rightarrow \pi/2} f(x)$

- (c) For what values of a does $\lim_{x \rightarrow a} f(x)$ exist?

- 55.** If $f(x) = \llbracket x \rrbracket + \llbracket -x \rrbracket$, show that $\lim_{x \rightarrow 2} f(x)$ exists but is not equal to $f(2)$.

- 56.** In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \rightarrow c^-} L$ and interpret the result. Why is a left-hand limit necessary?

- 57.** If p is a polynomial, show that $\lim_{x \rightarrow a} p(x) = p(a)$.

- 58.** If r is a rational function, use Exercise 57 to show that $\lim_{x \rightarrow a} r(x) = r(a)$ for every number a in the domain of r .

59. If $\lim_{x \rightarrow 1} \frac{f(x) - 8}{x - 1} = 10$, find $\lim_{x \rightarrow 1} f(x)$.

60. If $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 5$, find the following limits.

(a) $\lim_{x \rightarrow 0} f(x)$ (b) $\lim_{x \rightarrow 0} \frac{f(x)}{x}$

61. If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \rightarrow 0} f(x) = 0$.

62. Show by means of an example that $\lim_{x \rightarrow a} [f(x) + g(x)]$ may exist even though neither $\lim_{x \rightarrow a} f(x)$ nor $\lim_{x \rightarrow a} g(x)$ exists.

63. Show by means of an example that $\lim_{x \rightarrow a} [f(x)g(x)]$ may exist even though neither $\lim_{x \rightarrow a} f(x)$ nor $\lim_{x \rightarrow a} g(x)$ exists.

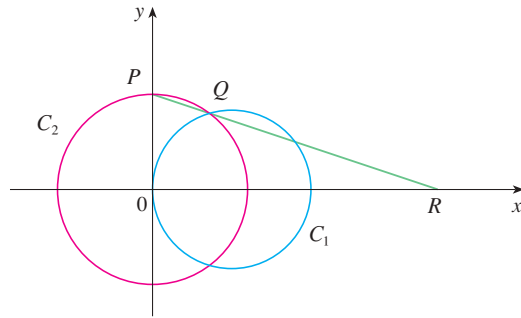
64. Evaluate $\lim_{x \rightarrow 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1}$.

65. Is there a number a such that

$$\lim_{x \rightarrow -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

66. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x -axis. What happens to R as C_2 shrinks, that is, as $r \rightarrow 0^+$?



1.7 The Precise Definition of a Limit

The intuitive definition of a limit given in Section 1.5 is inadequate for some purposes because such phrases as “ x is close to 2” and “ $f(x)$ gets closer and closer to L ” are vague. In order to be able to prove conclusively that

$$\lim_{x \rightarrow 0} \left(x^3 + \frac{\cos 5x}{10,000} \right) = 0.0001 \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

we must make the definition of a limit precise.

To motivate the precise definition of a limit, let's consider the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Intuitively, it is clear that when x is close to 3 but $x \neq 3$, then $f(x)$ is close to 5, and so $\lim_{x \rightarrow 3} f(x) = 5$.

To obtain more detailed information about how $f(x)$ varies when x is close to 3, we ask the following question:

How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.1?

The distance from x to 3 is $|x - 3|$ and the distance from $f(x)$ to 5 is $|f(x) - 5|$, so our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad |x - 3| < \delta \quad \text{but } x \neq 3$$

If $|x - 3| > 0$, then $x \neq 3$, so an equivalent formulation of our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad 0 < |x - 3| < \delta$$

It is traditional to use the Greek letter δ (delta) in this situation.

Notice that if $0 < |x - 3| < (0.1)/2 = 0.05$, then

$$|f(x) - 5| = |(2x - 1) - 5| = |2x - 6| = 2|x - 3| < 2(0.05) = 0.1$$

that is, $|f(x) - 5| < 0.1$ if $0 < |x - 3| < 0.05$

Thus an answer to the problem is given by $\delta = 0.05$; that is, if x is within a distance of 0.05 from 3, then $f(x)$ will be within a distance of 0.1 from 5.

If we change the number 0.1 in our problem to the smaller number 0.01, then by using the same method we find that $f(x)$ will differ from 5 by less than 0.01 provided that x differs from 3 by less than $(0.01)/2 = 0.005$:

$$|f(x) - 5| < 0.01 \quad \text{if} \quad 0 < |x - 3| < 0.005$$

Similarly,

$$|f(x) - 5| < 0.001 \quad \text{if} \quad 0 < |x - 3| < 0.0005$$

The numbers 0.1, 0.01, and 0.001 that we have considered are *error tolerances* that we might allow. For 5 to be the precise limit of $f(x)$ as x approaches 3, we must not only be able to bring the difference between $f(x)$ and 5 below each of these three numbers; we must be able to bring it below *any* positive number. And, by the same reasoning, we can! If we write ε (the Greek letter epsilon) for an arbitrary positive number, then we find as before that

$$\boxed{1} \quad |f(x) - 5| < \varepsilon \quad \text{if} \quad 0 < |x - 3| < \delta = \frac{\varepsilon}{2}$$

This is a precise way of saying that $f(x)$ is close to 5 when x is close to 3 because (1) says that we can make the values of $f(x)$ within an arbitrary distance ε from 5 by restricting the values of x to be within a distance $\varepsilon/2$ from 3 (but $x \neq 3$).

Note that (1) can be rewritten as follows:

$$\text{if } 3 - \delta < x < 3 + \delta \quad (x \neq 3) \quad \text{then} \quad 5 - \varepsilon < f(x) < 5 + \varepsilon$$

and this is illustrated in Figure 1. By taking the values of x ($\neq 3$) to lie in the interval $(3 - \delta, 3 + \delta)$ we can make the values of $f(x)$ lie in the interval $(5 - \varepsilon, 5 + \varepsilon)$.

Using (1) as a model, we give a precise definition of a limit.

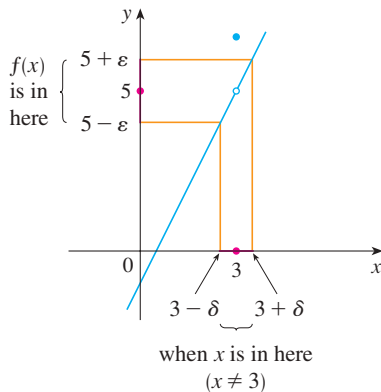


FIGURE 1

2 Precise Definition of a Limit Let f be a function defined on some open interval that contains the number a , except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L** , and we write

$$\lim_{x \rightarrow a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Since $|x - a|$ is the distance from x to a and $|f(x) - L|$ is the distance from $f(x)$ to L , and since ε can be arbitrarily small, the definition of a limit can be expressed in words as follows:

$\lim_{x \rightarrow a} f(x) = L$ means that the distance between $f(x)$ and L can be made arbitrarily small by requiring that the distance from x to a be sufficiently small (but not 0).

Alternatively,

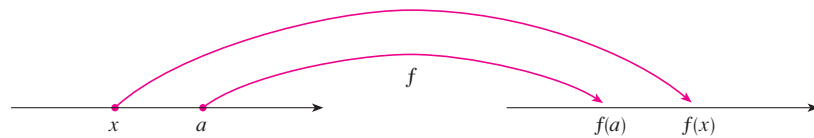
$\lim_{x \rightarrow a} f(x) = L$ means that the values of $f(x)$ can be made as close as we please to L by requiring x to be close enough to a (but not equal to a).

We can also reformulate Definition 2 in terms of intervals by observing that the inequality $|x - a| < \delta$ is equivalent to $-\delta < x - a < \delta$, which in turn can be written as $a - \delta < x < a + \delta$. Also $0 < |x - a|$ is true if and only if $x - a \neq 0$, that is, $x \neq a$. Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to the pair of inequalities $L - \varepsilon < f(x) < L + \varepsilon$. Therefore, in terms of intervals, Definition 2 can be stated as follows:

$\lim_{x \rightarrow a} f(x) = L$ means that for every $\varepsilon > 0$ (no matter how small ε is) we can find $\delta > 0$ such that if x lies in the open interval $(a - \delta, a + \delta)$ and $x \neq a$, then $f(x)$ lies in the open interval $(L - \varepsilon, L + \varepsilon)$.

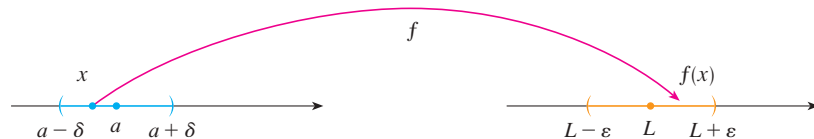
We interpret this statement geometrically by representing a function by an arrow diagram as in Figure 2, where f maps a subset of $\mathbb{R}$ onto another subset of $\mathbb{R}$.

FIGURE 2



The definition of limit says that if any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L , then we can find an interval $(a - \delta, a + \delta)$ around a such that f maps all the points in $(a - \delta, a + \delta)$ (except possibly a) into the interval $(L - \varepsilon, L + \varepsilon)$. (See Figure 3.)

FIGURE 3



Another geometric interpretation of limits can be given in terms of the graph of a function. If $\varepsilon > 0$ is given, then we draw the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ and the graph of f . (See Figure 4.) If $\lim_{x \rightarrow a} f(x) = L$, then we can find a number $\delta > 0$ such that if we restrict x to lie in the interval $(a - \delta, a + \delta)$ and take $x \neq a$, then the curve $y = f(x)$ lies between the lines $y = L - \varepsilon$ and $y = L + \varepsilon$. (See Figure 5.) You can see that if such a δ has been found, then any smaller δ will also work.

It is important to realize that the process illustrated in Figures 4 and 5 must work for *every* positive number ε , no matter how small it is chosen. Figure 6 shows that if a smaller ε is chosen, then a smaller δ may be required.

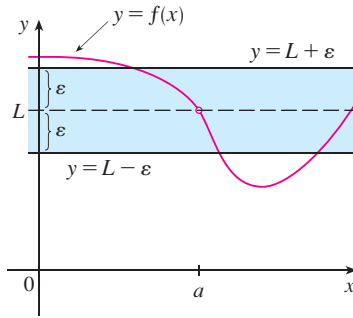


FIGURE 4

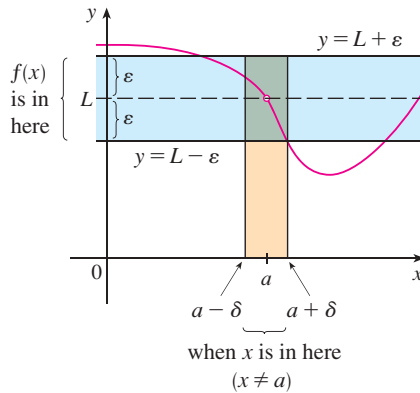


FIGURE 5

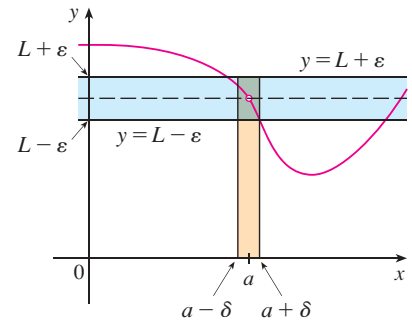


FIGURE 6

EXAMPLE 1 Since $f(x) = x^3 - 5x + 6$ is a polynomial, we know from the Direct Substitution Property that $\lim_{x \rightarrow 1} f(x) = f(1) = 1^3 - 5(1) + 6 = 2$. Use a graph to find a number δ such that if x is within δ of 1, then y is within 0.2 of 2, that is,

$$\text{if } |x - 1| < \delta \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.2$$

In other words, find a number δ that corresponds to $\varepsilon = 0.2$ in the definition of a limit for the function $f(x) = x^3 - 5x + 6$ with $a = 1$ and $L = 2$.

SOLUTION A graph of f is shown in Figure 7; we are interested in the region near the point $(1, 2)$. Notice that we can rewrite the inequality

$$|(x^3 - 5x + 6) - 2| < 0.2$$

$$\text{as} \quad -0.2 < (x^3 - 5x + 6) - 2 < 0.2$$

$$\text{or equivalently} \quad 1.8 < x^3 - 5x + 6 < 2.2$$

So we need to determine the values of x for which the curve $y = x^3 - 5x + 6$ lies between the horizontal lines $y = 1.8$ and $y = 2.2$. Therefore we graph the curves $y = x^3 - 5x + 6$, $y = 1.8$, and $y = 2.2$ near the point $(1, 2)$ in Figure 8. Then we use the cursor to estimate that the x -coordinate of the point of intersection of the line $y = 2.2$ and the curve $y = x^3 - 5x + 6$ is about 0.911. Similarly, $y = x^3 - 5x + 6$ intersects the line $y = 1.8$ when $x \approx 1.124$. So, rounding toward 1 to be safe, we can say that

$$\text{if } 0.92 < x < 1.12 \quad \text{then} \quad 1.8 < x^3 - 5x + 6 < 2.2$$

This interval $(0.92, 1.12)$ is not symmetric about $x = 1$. The distance from $x = 1$ to the left endpoint is $1 - 0.92 = 0.08$ and the distance to the right endpoint is 0.12. We can choose δ to be the smaller of these numbers, that is, $\delta = 0.08$. Then we can rewrite our inequalities in terms of distances as follows:

$$\text{if } |x - 1| < 0.08 \quad \text{then} \quad |(x^3 - 5x + 6) - 2| < 0.2$$

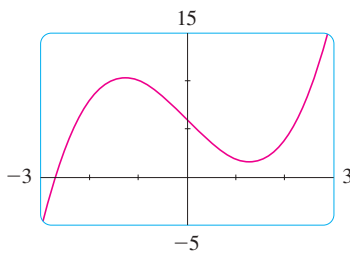


FIGURE 7

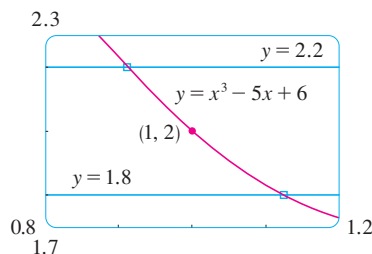


FIGURE 8

This just says that by keeping x within 0.08 of 1, we are able to keep $f(x)$ within 0.2 of 2.

Although we chose $\delta = 0.08$, any smaller positive value of δ would also have worked. ■

The graphical procedure in Example 1 gives an illustration of the definition for $\varepsilon = 0.2$, but it does not *prove* that the limit is equal to 2. A proof has to provide a δ for *every* ε .

In proving limit statements it may be helpful to think of the definition of limit as a challenge. First it challenges you with a number ε . Then you must be able to produce a suitable δ . You have to be able to do this for *every* $\varepsilon > 0$, not just a particular ε .

Imagine a contest between two people, A and B, and imagine yourself to be B. Person A stipulates that the fixed number L should be approximated by the values of $f(x)$ to within a degree of accuracy ε (say, 0.01). Person B then responds by finding a number δ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Then A may become more exacting and challenge B with a smaller value of ε (say, 0.0001). Again B has to respond by finding a corresponding δ . Usually the smaller the value of ε , the smaller the corresponding value of δ must be. If B always wins, no matter how small A makes ε , then $\lim_{x \rightarrow a} f(x) = L$.

TEC In Module 1.7/3.4 you can explore the precise definition of a limit both graphically and numerically.

EXAMPLE 2 Prove that $\lim_{x \rightarrow 3} (4x - 5) = 7$.

SOLUTION

1. Preliminary analysis of the problem (guessing a value for δ). Let ε be a given positive number. We want to find a number δ such that

$$\text{if } 0 < |x - 3| < \delta \quad \text{then} \quad |(4x - 5) - 7| < \varepsilon$$

But $|(4x - 5) - 7| = |4x - 12| = |4(x - 3)| = 4|x - 3|$. Therefore we want δ such that

$$\text{if } 0 < |x - 3| < \delta \quad \text{then} \quad 4|x - 3| < \varepsilon$$

$$\text{that is,} \quad \text{if } 0 < |x - 3| < \delta \quad \text{then} \quad |x - 3| < \frac{\varepsilon}{4}$$

This suggests that we should choose $\delta = \varepsilon/4$.

2. Proof (showing that this δ works). Given $\varepsilon > 0$, choose $\delta = \varepsilon/4$. If $0 < |x - 3| < \delta$, then

$$|(4x - 5) - 7| = |4x - 12| = 4|x - 3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$

Thus

$$\text{if } 0 < |x - 3| < \delta \quad \text{then} \quad |(4x - 5) - 7| < \varepsilon$$

Therefore, by the definition of a limit,

$$\lim_{x \rightarrow 3} (4x - 5) = 7$$

Cauchy and Limits

After the invention of calculus in the 17th century, there followed a period of free development of the subject in the 18th century. Mathematicians like the Bernoulli brothers and Euler were eager to exploit the power of calculus and boldly explored the consequences of this new and wonderful mathematical theory without worrying too much about whether their proofs were completely correct.

The 19th century, by contrast, was the Age of Rigor in mathematics. There was a movement to go back to the foundations of the subject—to provide careful definitions and rigorous proofs. At the forefront of this movement was the French mathematician Augustin-Louis Cauchy (1789–1857), who started out as a military engineer before becoming a mathematics professor in Paris. Cauchy took Newton's idea of a limit, which was kept alive in the 18th century by the French mathematician Jean d'Alembert, and made it more precise. His definition of a limit reads as follows: "When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the *limit* of all the others." But when Cauchy used this definition in examples and proofs, he often employed delta-epsilon inequalities similar to the ones in this section. A typical Cauchy proof starts with: "Designate by δ and ε two very small numbers; . . ." He used ε because of the correspondence between epsilon and the French word *erreur* and δ because delta corresponds to *différence*. Later, the German mathematician Karl Weierstrass (1815–1897) stated the definition of a limit exactly as in our Definition 2.

This example is illustrated by Figure 9.

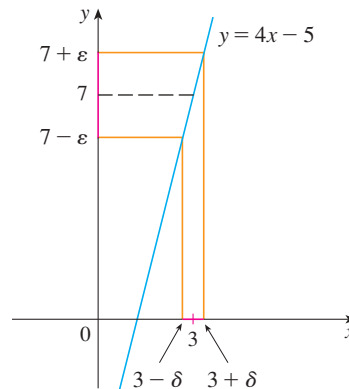


FIGURE 9

Note that in the solution of Example 2 there were two stages—guessing and proving. We made a preliminary analysis that enabled us to guess a value for δ . But then in the second stage we had to go back and prove in a careful, logical fashion that we had made a correct guess. This procedure is typical of much of mathematics. Sometimes it is necessary to first make an intelligent guess about the answer to a problem and then prove that the guess is correct.

The intuitive definitions of one-sided limits that were given in Section 1.5 can be precisely reformulated as follows.

3 Definition of Left-Hand Limit

$$\lim_{x \rightarrow a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \varepsilon$$

4 Definition of Right-Hand Limit

$$\lim_{x \rightarrow a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Notice that Definition 3 is the same as Definition 2 except that x is restricted to lie in the *left* half $(a - \delta, a)$ of the interval $(a - \delta, a + \delta)$. In Definition 4, x is restricted to lie in the *right* half $(a, a + \delta)$ of the interval $(a - \delta, a + \delta)$.

EXAMPLE 3 Use Definition 4 to prove that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$.

SOLUTION

1. *Guessing a value for δ .* Let ε be a given positive number. Here $a = 0$ and $L = 0$, so we want to find a number δ such that

$$\text{if } 0 < x < \delta \quad \text{then} \quad |\sqrt{x} - 0| < \varepsilon$$

that is, if $0 < x < \delta$ then $\sqrt{x} < \varepsilon$

or, squaring both sides of the inequality $\sqrt{x} < \varepsilon$, we get

$$\text{if } 0 < x < \delta \quad \text{then} \quad x < \varepsilon^2$$

This suggests that we should choose $\delta = \varepsilon^2$.

2. Showing that this δ works. Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. If $0 < x < \delta$, then

$$\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

$$\text{so} \quad |\sqrt{x} - 0| < \varepsilon$$

According to Definition 4, this shows that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$. ■

EXAMPLE 4 Prove that $\lim_{x \rightarrow 3} x^2 = 9$.

SOLUTION

1. Guessing a value for δ . Let $\varepsilon > 0$ be given. We have to find a number $\delta > 0$ such that

$$\text{if } 0 < |x - 3| < \delta \quad \text{then} \quad |x^2 - 9| < \varepsilon$$

To connect $|x^2 - 9|$ with $|x - 3|$ we write $|x^2 - 9| = |(x + 3)(x - 3)|$. Then we want

$$\text{if } 0 < |x - 3| < \delta \quad \text{then} \quad |x + 3||x - 3| < \varepsilon$$

Notice that if we can find a positive constant C such that $|x + 3| < C$, then

$$|x + 3||x - 3| < C|x - 3|$$

and we can make $C|x - 3| < \varepsilon$ by taking $|x - 3| < \varepsilon/C$, so we could choose $\delta = \varepsilon/C$.

We can find such a number C if we restrict x to lie in some interval centered at 3. In fact, since we are interested only in values of x that are close to 3, it is reasonable to assume that x is within a distance 1 from 3, that is, $|x - 3| < 1$. Then $2 < x < 4$, so $5 < x + 3 < 7$. Thus we have $|x + 3| < 7$, and so $C = 7$ is a suitable choice for the constant.

But now there are two restrictions on $|x - 3|$, namely

$$|x - 3| < 1 \quad \text{and} \quad |x - 3| < \frac{\varepsilon}{C} = \frac{\varepsilon}{7}$$

To make sure that both of these inequalities are satisfied, we take δ to be the smaller of the two numbers 1 and $\varepsilon/7$. The notation for this is $\delta = \min\{1, \varepsilon/7\}$.

2. Showing that this δ works. Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/7\}$. If $0 < |x - 3| < \delta$, then $|x - 3| < 1 \Rightarrow 2 < x < 4 \Rightarrow |x + 3| < 7$ (as in part 1).

We also have $|x - 3| < \varepsilon/7$, so

$$|x^2 - 9| = |x + 3||x - 3| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon$$

This shows that $\lim_{x \rightarrow 3} x^2 = 9$. ■

As Example 4 shows, it is not always easy to prove that limit statements are true using the ε, δ definition. In fact, if we had been given a more complicated function such as $f(x) = (6x^2 - 8x + 9)/(2x^2 - 1)$, a proof would require a great deal of ingenuity. Fortunately this is unnecessary because the Limit Laws stated in Section 1.6 can be proved using Definition 2, and then the limits of complicated functions can be found rigorously from the Limit Laws without resorting to the definition directly.

For instance, we prove the Sum Law: If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$ both exist, then

$$\lim_{x \rightarrow a} [f(x) + g(x)] = L + M$$

The remaining laws are proved in the exercises and in Appendix F.

PROOF OF THE SUM LAW Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |f(x) + g(x) - (L + M)| < \varepsilon$$

Triangle Inequality:

$$|a + b| \leq |a| + |b|$$

(See Appendix A).

Using the Triangle Inequality we can write

$$\begin{aligned} \boxed{5} \quad |f(x) + g(x) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \end{aligned}$$

We make $|f(x) + g(x) - (L + M)|$ less than ε by making each of the terms $|f(x) - L|$ and $|g(x) - M|$ less than $\varepsilon/2$.

Since $\varepsilon/2 > 0$ and $\lim_{x \rightarrow a} f(x) = L$, there exists a number $\delta_1 > 0$ such that

$$\text{if } 0 < |x - a| < \delta_1 \quad \text{then} \quad |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, since $\lim_{x \rightarrow a} g(x) = M$, there exists a number $\delta_2 > 0$ such that

$$\text{if } 0 < |x - a| < \delta_2 \quad \text{then} \quad |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of the numbers δ_1 and δ_2 . Notice that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad 0 < |x - a| < \delta_1 \quad \text{and} \quad 0 < |x - a| < \delta_2$$

$$\text{and so} \quad |f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(x) - M| < \frac{\varepsilon}{2}$$

Therefore, by (5),

$$\begin{aligned} |f(x) + g(x) - (L + M)| &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

To summarize,

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |f(x) + g(x) - (L + M)| < \varepsilon$$

Thus, by the definition of a limit,

$$\lim_{x \rightarrow a} [f(x) + g(x)] = L + M$$

■ Infinite Limits

Infinite limits can also be defined in a precise way. The following is a precise version of Definition 1.5.4.

6 Precise Definition of an Infinite Limit Let f be a function defined on some open interval that contains the number a , except possibly at a itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M$$

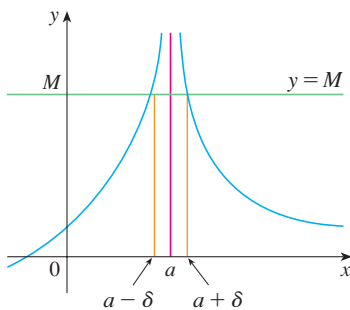


FIGURE 10

This says that the values of $f(x)$ can be made arbitrarily large (larger than any given number M) by requiring x to be close enough to a (within a distance δ , where δ depends on M , but with $x \neq a$). A geometric illustration is shown in Figure 10.

Given any horizontal line $y = M$, we can find a number $\delta > 0$ such that if we restrict x to lie in the interval $(a - \delta, a + \delta)$ but $x \neq a$, then the curve $y = f(x)$ lies above the line $y = M$. You can see that if a larger M is chosen, then a smaller δ may be required.

EXAMPLE 5 Use Definition 6 to prove that $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$.

SOLUTION Let M be a given positive number. We want to find a number δ such that

$$\text{if } 0 < |x - 0| < \delta \quad \text{then} \quad 1/x^2 > M$$

$$\text{But } \frac{1}{x^2} > M \iff x^2 < \frac{1}{M} \iff \sqrt{x^2} < \sqrt{\frac{1}{M}} \iff |x| < \frac{1}{\sqrt{M}}$$

So if we choose $\delta = 1/\sqrt{M}$ and $0 < |x| < \delta = 1/\sqrt{M}$, then $1/x^2 > M$. This shows that $1/x^2 \rightarrow \infty$ as $x \rightarrow 0$.

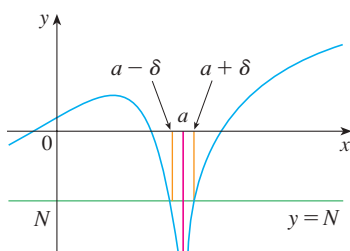


FIGURE 11

Similarly, the following is a precise version of Definition 1.5.5. It is illustrated by Figure 11.

7 Definition Let f be a function defined on some open interval that contains the number a , except possibly at a itself. Then

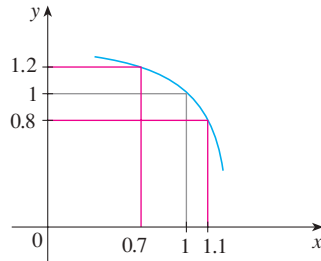
$$\lim_{x \rightarrow a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that

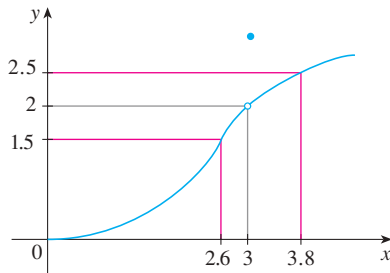
$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad f(x) < N$$

1.7 EXERCISES

1. Use the given graph of f to find a number δ such that
if $|x - 1| < \delta$ then $|f(x) - 1| < 0.2$

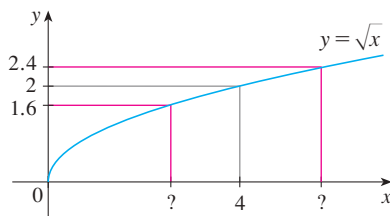


2. Use the given graph of f to find a number δ such that
if $0 < |x - 3| < \delta$ then $|f(x) - 2| < 0.5$



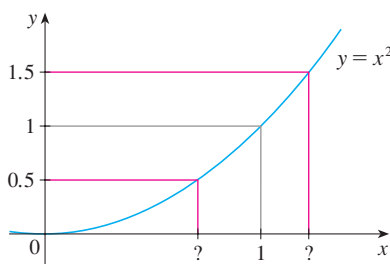
3. Use the given graph of $f(x) = \sqrt{x}$ to find a number δ such that

if $|x - 4| < \delta$ then $|\sqrt{x} - 2| < 0.4$



4. Use the given graph of $f(x) = x^2$ to find a number δ such that

if $|x - 1| < \delta$ then $|x^2 - 1| < \frac{1}{2}$



5. Use a graph to find a number δ such that

if $\left|x - \frac{\pi}{4}\right| < \delta$ then $|\tan x - 1| < 0.2$

6. Use a graph to find a number δ such that

if $|x - 1| < \delta$ then $\left|\frac{2x}{x^2 + 4} - 0.4\right| < 0.1$

7. For the limit

$$\lim_{x \rightarrow 2} (x^3 - 3x + 4) = 6$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.2$ and $\varepsilon = 0.1$.

8. For the limit

$$\lim_{x \rightarrow 2} \frac{4x + 1}{3x - 4} = 4.5$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

9. (a) Use a graph to find a number δ such that

if $4 < x < 4 + \delta$ then $\frac{x^2 + 4}{\sqrt{x} - 4} > 100$

(b) What limit does part (a) suggest is true?

10. Given that $\lim_{x \rightarrow \pi} \csc^2 x = \infty$, illustrate Definition 6 by finding values of δ that correspond to (a) $M = 500$ and (b) $M = 1000$.

11. A machinist is required to manufacture a circular metal disk with area 1000 cm^2 .

- (a) What radius produces such a disk?
(b) If the machinist is allowed an error tolerance of $\pm 5 \text{ cm}^2$ in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius?
(c) In terms of the ε, δ definition of $\lim_{x \rightarrow a} f(x) = L$, what is x ? What is $f(x)$? What is a ? What is L ? What value of ε is given? What is the corresponding value of δ ?

12. A crystal growth furnace is used in research to determine how best to manufacture crystals used in electronic components for the space shuttle. For proper growth of the crystal, the temperature must be controlled accurately by adjusting the input power. Suppose the relationship is given by

$$T(w) = 0.1w^2 + 2.155w + 20$$

where T is the temperature in degrees Celsius and w is the power input in watts.

- (a) How much power is needed to maintain the temperature at 200°C ?

- (b) If the temperature is allowed to vary from 200°C by up to $\pm 1^{\circ}\text{C}$, what range of wattage is allowed for the input power?
- (c) In terms of the ε, δ definition of $\lim_{x \rightarrow a} f(x) = L$, what is x ? What is $f(x)$? What is a ? What is L ? What value of ε is given? What is the corresponding value of δ ?
13. (a) Find a number δ such that if $|x - 2| < \delta$, then $|4x - 8| < \varepsilon$, where $\varepsilon = 0.1$.
(b) Repeat part (a) with $\varepsilon = 0.01$.
14. Given that $\lim_{x \rightarrow 2} (5x - 7) = 3$, illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.1$, $\varepsilon = 0.05$, and $\varepsilon = 0.01$.

15–18 Prove the statement using the ε, δ definition of a limit and illustrate with a diagram like Figure 9.

15. $\lim_{x \rightarrow 3} (1 + \frac{1}{3}x) = 2$ 16. $\lim_{x \rightarrow 4} (2x - 5) = 3$
 17. $\lim_{x \rightarrow -3} (1 - 4x) = 13$ 18. $\lim_{x \rightarrow -2} (3x + 5) = -1$

19–32 Prove the statement using the ε, δ definition of a limit.

19. $\lim_{x \rightarrow 1} \frac{2 + 4x}{3} = 2$ 20. $\lim_{x \rightarrow 10} (3 - \frac{4}{5}x) = -5$
 21. $\lim_{x \rightarrow 4} \frac{x^2 - 2x - 8}{x - 4} = 6$ 22. $\lim_{x \rightarrow -1.5} \frac{9 - 4x^2}{3 + 2x} = 6$
 23. $\lim_{x \rightarrow a} x = a$ 24. $\lim_{x \rightarrow a} c = c$
 25. $\lim_{x \rightarrow 0} x^2 = 0$ 26. $\lim_{x \rightarrow 0} x^3 = 0$
 27. $\lim_{x \rightarrow 0} |x| = 0$ 28. $\lim_{x \rightarrow -6^+} \sqrt[8]{6 + x} = 0$
 29. $\lim_{x \rightarrow 2} (x^2 - 4x + 5) = 1$ 30. $\lim_{x \rightarrow 2} (x^2 + 2x - 7) = 1$
 31. $\lim_{x \rightarrow -2} (x^2 - 1) = 3$ 32. $\lim_{x \rightarrow 2} x^3 = 8$

33. Verify that another possible choice of δ for showing that $\lim_{x \rightarrow 3} x^2 = 9$ in Example 4 is $\delta = \min\{2, \varepsilon/8\}$.
34. Verify, by a geometric argument, that the largest possible choice of δ for showing that $\lim_{x \rightarrow 3} x^2 = 9$ is $\delta = \sqrt{9 + \varepsilon} - 3$.

- CAS** 35. (a) For the limit $\lim_{x \rightarrow 1} (x^3 + x + 1) = 3$, use a graph to find a value of δ that corresponds to $\varepsilon = 0.4$.
 (b) By using a computer algebra system to solve the cubic equation $x^3 + x + 1 = 3 + \varepsilon$, find the largest possible value of δ that works for any given $\varepsilon > 0$.
 (c) Put $\varepsilon = 0.4$ in your answer to part (b) and compare with your answer to part (a).

36. Prove that $\lim_{x \rightarrow 2} \frac{1}{x} = \frac{1}{2}$.

37. Prove that $\lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}$ if $a > 0$.

[Hint: Use $|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}}.$]

38. If H is the Heaviside function defined in Example 1.5.6, prove, using Definition 2, that $\lim_{t \rightarrow 0} H(t)$ does not exist. [Hint: Use an indirect proof as follows. Suppose that the limit is L . Take $\varepsilon = \frac{1}{2}$ in the definition of a limit and try to arrive at a contradiction.]

39. If the function f is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \rightarrow 0} f(x)$ does not exist.

40. By comparing Definitions 2, 3, and 4, prove Theorem 1.6.1.
41. How close to -3 do we have to take x so that

$$\frac{1}{(x + 3)^4} > 10,000$$

42. Prove, using Definition 6, that $\lim_{x \rightarrow -3} \frac{1}{(x + 3)^4} = \infty$.

43. Prove that $\lim_{x \rightarrow -1^-} \frac{5}{(x + 1)^3} = -\infty$.

44. Suppose that $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = c$, where c is a real number. Prove each statement.

- (a) $\lim_{x \rightarrow a} [f(x) + g(x)] = \infty$
 (b) $\lim_{x \rightarrow a} [f(x)g(x)] = \infty$ if $c > 0$
 (c) $\lim_{x \rightarrow a} [f(x)g(x)] = -\infty$ if $c < 0$

1.8 Continuity

We noticed in Section 1.6 that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a . Functions with this property are called *continuous at a*. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

1 Definition A function f is **continuous at a number a** if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

As illustrated in Figure 1, if f is continuous, then the points $(x, f(x))$ on the graph of f approach the point $(a, f(a))$ on the graph. So there is no gap in the curve.

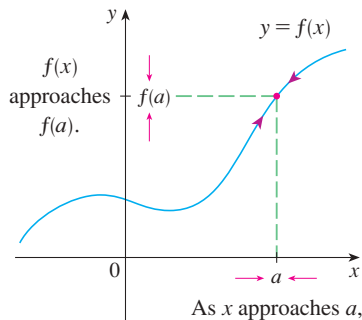


FIGURE 1

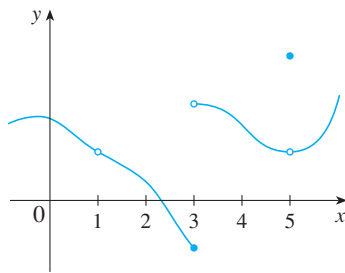


FIGURE 2

Notice that Definition 1 implicitly requires three things if f is continuous at a :

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \rightarrow a} f(x)$ exists
3. $\lim_{x \rightarrow a} f(x) = f(a)$

The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a . Thus a continuous function f has the property that a small change in x produces only a small change in $f(x)$. In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near a (in other words, f is defined on an open interval containing a , except perhaps at a), we say that f is **discontinuous at a** (or f has a **discontinuity at a**) if f is not continuous at a .

Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents. [See Example 1.5.6, where the Heaviside function is discontinuous at 0 because $\lim_{t \rightarrow 0} H(t)$ does not exist.]

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it: the graph can be drawn without removing your pen from the paper.

EXAMPLE 1 Figure 2 shows the graph of a function f . At which numbers is f discontinuous? Why?

SOLUTION It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The official reason that f is discontinuous at 1 is that $f(1)$ is not defined.

The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x \rightarrow 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3.

What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x \rightarrow 5} f(x)$ exists (because the left and right limits are the same). But

$$\lim_{x \rightarrow 5} f(x) \neq f(5)$$

So f is discontinuous at 5. ■

Now let's see how to detect discontinuities when a function is defined by a formula.

EXAMPLE 2 Where are each of the following functions discontinuous?

$$\begin{aligned} \text{(a)} \quad f(x) &= \frac{x^2 - x - 2}{x - 2} & \text{(b)} \quad f(x) &= \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \\ \text{(c)} \quad f(x) &= \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases} & \text{(d)} \quad f(x) &= \llbracket x \rrbracket \end{aligned}$$

SOLUTION

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why f is continuous at all other numbers.

(b) Here $f(0) = 1$ is defined but

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{1}{x^2}$$

does not exist. (See Example 1.5.8.) So f is discontinuous at 0.

(c) Here $f(2) = 1$ is defined and

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x \rightarrow 2} (x + 1) = 3$$

exists. But

$$\lim_{x \rightarrow 2} f(x) \neq f(2)$$

so f is not continuous at 2.

(d) The greatest integer function $f(x) = \llbracket x \rrbracket$ has discontinuities at all of the integers because $\lim_{x \rightarrow n} \llbracket x \rrbracket$ does not exist if n is an integer. (See Example 1.6.10 and Exercise 1.6.53.)

Figure 3 shows the graphs of the functions in Example 2. In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.] The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function “jumps” from one value to another.

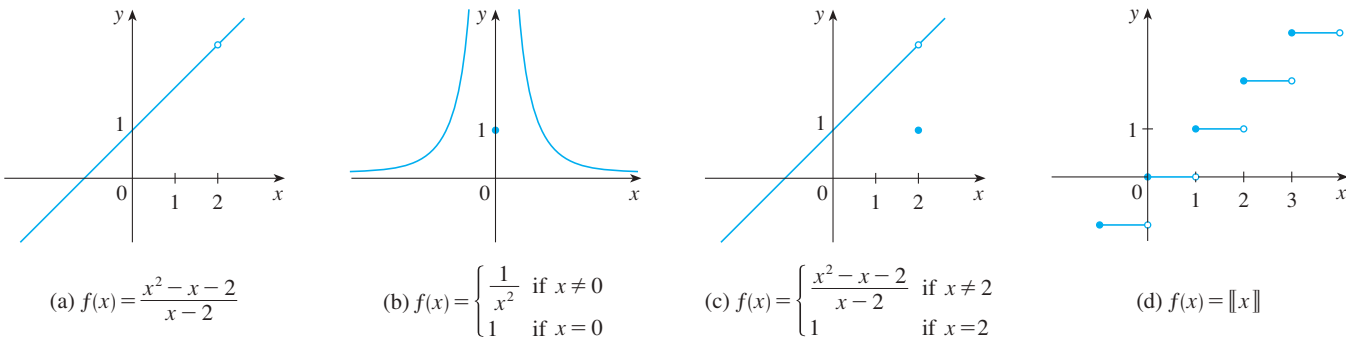


FIGURE 3

Graphs of the functions in Example 2

2 Definition A function f is **continuous from the right at a number a** if

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \rightarrow a^-} f(x) = f(a)$$

EXAMPLE 3 At each integer n , the function $f(x) = \llbracket x \rrbracket$ [see Figure 3(d)] is continuous from the right but discontinuous from the left because

$$\lim_{x \rightarrow n^+} f(x) = \lim_{x \rightarrow n^+} \llbracket x \rrbracket = n = f(n)$$

but

$$\lim_{x \rightarrow n^-} f(x) = \lim_{x \rightarrow n^-} \llbracket x \rrbracket = n - 1 \neq f(n)$$

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

EXAMPLE 4 Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

SOLUTION If $-1 < a < 1$, then using the Limit Laws, we have

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} (1 - \sqrt{1 - x^2}) \\ &= 1 - \lim_{x \rightarrow a} \sqrt{1 - x^2} && \text{(by Laws 2 and 7)} \\ &= 1 - \sqrt{\lim_{x \rightarrow a} (1 - x^2)} && \text{(by 11)} \\ &= 1 - \sqrt{1 - a^2} && \text{(by 2, 7, and 9)} \\ &= f(a) \end{aligned}$$

Thus, by Definition 1, f is continuous at a if $-1 < a < 1$. Similar calculations show that

$$\lim_{x \rightarrow -1^+} f(x) = 1 = f(-1) \quad \text{and} \quad \lim_{x \rightarrow 1^-} f(x) = 1 = f(1)$$

so f is continuous from the right at -1 and continuous from the left at 1 . Therefore, according to Definition 3, f is continuous on $[-1, 1]$.

The graph of f is sketched in Figure 4. It is the lower half of the circle

$$x^2 + (y - 1)^2 = 1$$

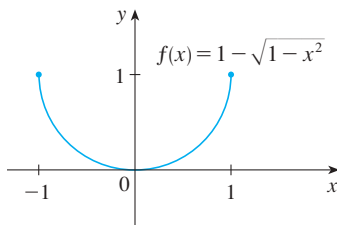


FIGURE 4

Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function as we did in Example 4, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones.

4 Theorem If f and g are continuous at a and if c is a constant, then the following functions are also continuous at a :

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$ if $g(a) \neq 0$

PROOF Each of the five parts of this theorem follows from the corresponding Limit Law in Section 1.6. For instance, we give the proof of part 1. Since f and g are continuous at a , we have

$$\lim_{x \rightarrow a} f(x) = f(a) \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = g(a)$$

Therefore

$$\begin{aligned}
 \lim_{x \rightarrow a} (f + g)(x) &= \lim_{x \rightarrow a} [f(x) + g(x)] \\
 &= \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) \quad (\text{by Law 1}) \\
 &= f(a) + g(a) \\
 &= (f + g)(a)
 \end{aligned}$$

This shows that $f + g$ is continuous at a . ■

It follows from Theorem 4 and Definition 3 that if f and g are continuous on an interval, then so are the functions $f + g$, $f - g$, cf , fg , and (if g is never 0) f/g . The following theorem was stated in Section 1.6 as the Direct Substitution Property.

5 Theorem

- (a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.
- (b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

PROOF

(a) A polynomial is a function of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \dots, c_n$ are constants. We know that

$$\lim_{x \rightarrow a} c_0 = c_0 \quad (\text{by Law 7})$$

and

$$\lim_{x \rightarrow a} x^m = a^m \quad m = 1, 2, \dots, n \quad (\text{by 9})$$

This equation is precisely the statement that the function $f(x) = x^m$ is a continuous function. Thus, by part 3 of Theorem 4, the function $g(x) = cx^m$ is continuous. Since P is a sum of functions of this form and a constant function, it follows from part 1 of Theorem 4 that P is continuous.

(b) A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain of f is $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$. We know from part (a) that P and Q are continuous everywhere. Thus, by part 5 of Theorem 4, f is continuous at every number in D . ■

As an illustration of Theorem 5, observe that the volume of a sphere varies continuously with its radius because the formula $V(r) = \frac{4}{3}\pi r^3$ shows that V is a polynomial function of r . Likewise, if a ball is thrown vertically into the air with a velocity of 50 ft/s, then the height of the ball in feet t seconds later is given by the formula $h = 50t - 16t^2$. Again this is a polynomial function, so the height is a continuous function of the elapsed time, as we might expect.

Knowledge of which functions are continuous enables us to evaluate some limits very quickly, as the following example shows. Compare it with Example 1.6.2(b).

EXAMPLE 5 Find $\lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$.

SOLUTION The function

$$f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

is rational, so by Theorem 5 it is continuous on its domain, which is $\{x \mid x \neq \frac{5}{3}\}$. Therefore

$$\begin{aligned} \lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} &= \lim_{x \rightarrow -2} f(x) = f(-2) \\ &= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11} \end{aligned}$$

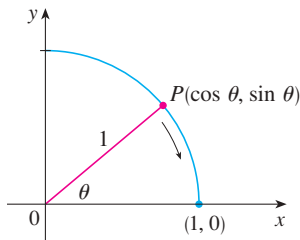


FIGURE 5

It turns out that most of the familiar functions are continuous at every number in their domains. For instance, Limit Law 10 (page 64) is exactly the statement that root functions are continuous.

From the appearance of the graphs of the sine and cosine functions (Figure 1.2.18), we would certainly guess that they are continuous. We know from the definitions of $\sin \theta$ and $\cos \theta$ that the coordinates of the point P in Figure 5 are $(\cos \theta, \sin \theta)$. As $\theta \rightarrow 0$, we see that P approaches the point $(1, 0)$ and so $\cos \theta \rightarrow 1$ and $\sin \theta \rightarrow 0$. Thus

$$\boxed{6} \quad \lim_{\theta \rightarrow 0} \cos \theta = 1 \quad \lim_{\theta \rightarrow 0} \sin \theta = 0$$

Another way to establish the limits in (6) is to use the Squeeze Theorem with the inequality $\sin \theta < \theta$ (for $\theta > 0$), which is proved in Section 2.4.

Since $\cos 0 = 1$ and $\sin 0 = 0$, the equations in (6) assert that the cosine and sine functions are continuous at 0. The addition formulas for cosine and sine can then be used to deduce that these functions are continuous everywhere (see Exercises 64 and 65).

It follows from part 5 of Theorem 4 that

$$\tan x = \frac{\sin x}{\cos x}$$

is continuous except where $\cos x = 0$. This happens when x is an odd integer multiple of $\pi/2$, so $y = \tan x$ has infinite discontinuities when $x = \pm\pi/2, \pm3\pi/2, \pm5\pi/2$, and so on (see Figure 6).

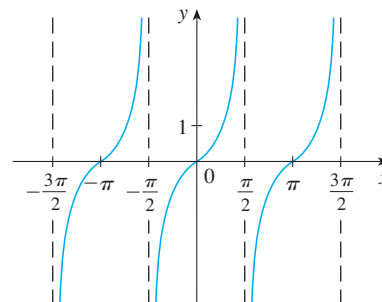


FIGURE 6
 $y = \tan x$

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions

EXAMPLE 6 On what intervals is each function continuous?

$$(a) f(x) = x^{100} - 2x^{37} + 75 \qquad (b) g(x) = \frac{x^2 + 2x + 17}{x^2 - 1}$$

$$(c) h(x) = \sqrt{x} + \frac{x+1}{x-1} - \frac{x+1}{x^2+1}$$

SOLUTION

(a) f is a polynomial, so it is continuous on $(-\infty, \infty)$ by Theorem 5(a).

(b) g is a rational function, so by Theorem 5(b), it is continuous on its domain, which is $D = \{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\}$. Thus g is continuous on the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

(c) We can write $h(x) = F(x) + G(x) - H(x)$, where

$$F(x) = \sqrt{x} \qquad G(x) = \frac{x+1}{x-1} \qquad H(x) = \frac{x+1}{x^2+1}$$

F is continuous on $[0, \infty)$ by Theorem 7. G is a rational function, so it is continuous everywhere except when $x - 1 = 0$, that is, $x = 1$. H is also a rational function, but its denominator is never 0, so H is continuous everywhere. Thus, by parts 1 and 2 of Theorem 4, h is continuous on the intervals $[0, 1)$ and $(1, \infty)$. ■

EXAMPLE 7 Evaluate $\lim_{x \rightarrow \pi} \frac{\sin x}{2 + \cos x}$.

SOLUTION Theorem 7 tells us that $y = \sin x$ is continuous. The function in the denominator, $y = 2 + \cos x$, is the sum of two continuous functions and is therefore continuous. Notice that this function is never 0 because $\cos x \geq -1$ for all x and so $2 + \cos x > 0$ everywhere. Thus the ratio

$$f(x) = \frac{\sin x}{2 + \cos x}$$

is continuous everywhere. Hence, by the definition of a continuous function,

$$\lim_{x \rightarrow \pi} \frac{\sin x}{2 + \cos x} = \lim_{x \rightarrow \pi} f(x) = f(\pi) = \frac{\sin \pi}{2 + \cos \pi} = \frac{0}{2 - 1} = 0 \quad \blacksquare$$

Another way of combining continuous functions f and g to get a new continuous function is to form the composite function $f \circ g$. This fact is a consequence of the following theorem.

8 Theorem If f is continuous at b and $\lim_{x \rightarrow a} g(x) = b$, then $\lim_{x \rightarrow a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right)$$

This theorem says that a limit symbol can be moved through a function symbol if the function is continuous and the limit exists. In other words, the order of these two symbols can be reversed.

Intuitively, Theorem 8 is reasonable because if x is close to a , then $g(x)$ is close to b , and since f is continuous at b , if $g(x)$ is close to b , then $f(g(x))$ is close to $f(b)$. A proof of Theorem 8 is given in Appendix F.

Let's now apply Theorem 8 in the special case where $f(x) = \sqrt[n]{x}$, with n being a positive integer. Then

$$f(g(x)) = \sqrt[n]{g(x)}$$

and

$$f\left(\lim_{x \rightarrow a} g(x)\right) = \sqrt[n]{\lim_{x \rightarrow a} g(x)}$$

If we put these expressions into Theorem 8, we get

$$\lim_{x \rightarrow a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x \rightarrow a} g(x)}$$

and so Limit Law 11 has now been proved. (We assume that the roots exist.)

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a .

This theorem is often expressed informally by saying “a continuous function of a continuous function is a continuous function.”

PROOF Since g is continuous at a , we have

$$\lim_{x \rightarrow a} g(x) = g(a)$$

Since f is continuous at $b = g(a)$, we can apply Theorem 8 to obtain

$$\lim_{x \rightarrow a} f(g(x)) = f(g(a))$$

which is precisely the statement that the function $h(x) = f(g(x))$ is continuous at a ; that is, $f \circ g$ is continuous at a . ■

EXAMPLE 8 Where are the following functions continuous?

$$(a) h(x) = \sin(x^2) \qquad (b) F(x) = \frac{1}{\sqrt{x^2 + 7} - 4}$$

SOLUTION

(a) We have $h(x) = f(g(x))$, where

$$g(x) = x^2 \qquad \text{and} \qquad f(x) = \sin x$$

Now g is continuous on $\mathbb{R}$ since it is a polynomial, and f is also continuous everywhere. Thus $h = f \circ g$ is continuous on $\mathbb{R}$ by Theorem 9.

(b) Notice that F can be broken up as the composition of four continuous functions:

$$F = f \circ g \circ h \circ k \qquad \text{or} \qquad F(x) = f(g(h(k(x))))$$

where $f(x) = \frac{1}{x}$ $g(x) = x - 4$ $h(x) = \sqrt{x}$ $k(x) = x^2 + 7$

We know that each of these functions is continuous on its domain (by Theorems 5 and 7), so by Theorem 9, F is continuous on its domain, which is

$$\{x \in \mathbb{R} \mid \sqrt{x^2 + 7} \neq 4\} = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty) \quad \blacksquare$$

An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 7. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].

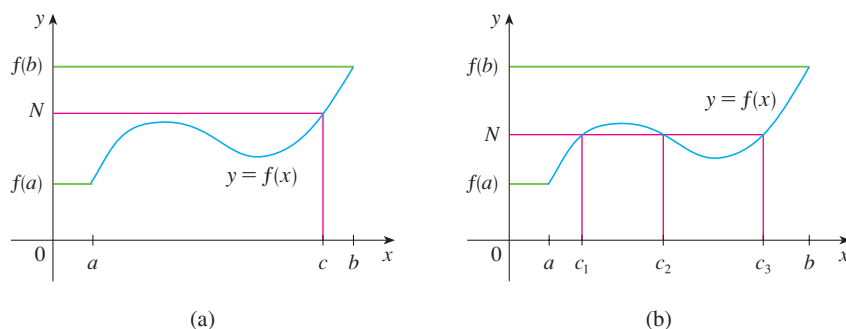


FIGURE 7

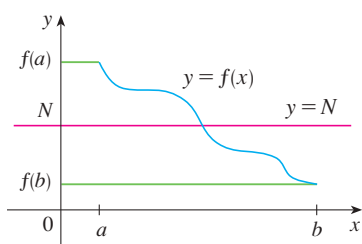


FIGURE 8

If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true. In geometric terms it says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ as in Figure 8, then the graph of f can't jump over the line. It must intersect $y = N$ somewhere.

It is important that the function f in Theorem 10 be continuous. The Intermediate Value Theorem is not true in general for discontinuous functions (see Exercise 50).

One use of the Intermediate Value Theorem is in locating roots of equations as in the following example.

EXAMPLE 9 Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

SOLUTION Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that $f(c) = 0$. Therefore we take $a = 1$, $b = 2$, and $N = 0$ in Theorem 10. We have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

and

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Thus $f(1) < 0 < f(2)$; that is, $N = 0$ is a number between $f(1)$ and $f(2)$. Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that $f(c) = 0$. In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval $(1, 2)$.

In fact, we can locate a root more precisely by using the Intermediate Value Theorem again. Since

$$f(1.2) = -0.128 < 0 \quad \text{and} \quad f(1.3) = 0.548 > 0$$

a root must lie between 1.2 and 1.3. A calculator gives, by trial and error,

$$f(1.22) = -0.007008 < 0 \quad \text{and} \quad f(1.23) = 0.056068 > 0$$

so a root lies in the interval $(1.22, 1.23)$. ■

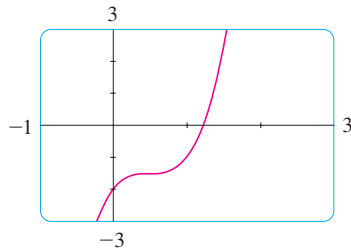


FIGURE 9

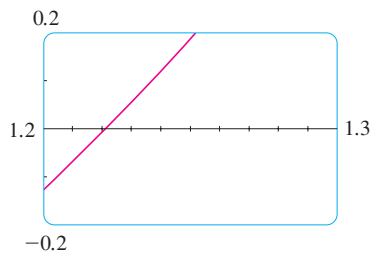


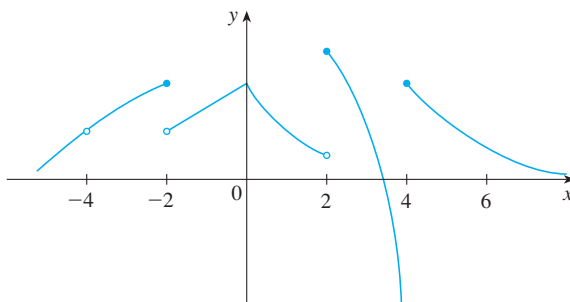
FIGURE 10

We can use a graphing calculator or computer to illustrate the use of the Intermediate Value Theorem in Example 9. Figure 9 shows the graph of f in the viewing rectangle $[-1, 3]$ by $[-3, 3]$ and you can see that the graph crosses the x -axis between 1 and 2. Figure 10 shows the result of zooming in to the viewing rectangle $[1.2, 1.3]$ by $[-0.2, 0.2]$.

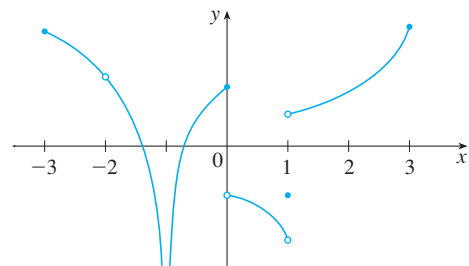
In fact, the Intermediate Value Theorem plays a role in the very way these graphing devices work. A computer calculates a finite number of points on the graph and turns on the pixels that contain these calculated points. It assumes that the function is continuous and takes on all the intermediate values between two consecutive points. The computer therefore “connects the dots” by turning on the intermediate pixels.

1.8 EXERCISES

- Write an equation that expresses the fact that a function f is continuous at the number 4.
- If f is continuous on $(-\infty, \infty)$, what can you say about its graph?
- From the graph of f , state the numbers at which f is discontinuous and explain why.
 - For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



- From the graph of g , state the intervals on which g is continuous.



- Sketch the graph of a function f that is continuous except for the stated discontinuity.

- Discontinuous at 2, but continuous from the right there
- Discontinuities at -1 and 4 , but continuous from the left at -1 and from the right at 4
- Removable discontinuity at 3, jump discontinuity at 5

8. Neither left nor right continuous at -2 , continuous only from the left at 2

9. The toll T charged for driving on a certain stretch of a toll road is \$5 except during rush hours (between 7 AM and 10 AM and between 4 PM and 7 PM) when the toll is \$7.

- (a) Sketch a graph of T as a function of the time t , measured in hours past midnight.
(b) Discuss the discontinuities of this function and their significance to someone who uses the road.

10. Explain why each function is continuous or discontinuous.

- (a) The temperature at a specific location as a function of time
(b) The temperature at a specific time as a function of the distance due west from New York City
(c) The altitude above sea level as a function of the distance due west from New York City
(d) The cost of a taxi ride as a function of the distance traveled
(e) The current in the circuit for the lights in a room as a function of time

11–14 Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a .

11. $f(x) = (x + 2x^3)^4$, $a = -1$

12. $g(t) = \frac{t^2 + 5t}{2t + 1}$, $a = 2$

13. $p(v) = 2\sqrt{3v^2 + 1}$, $a = 1$

14. $f(x) = 3x^4 - 5x + \sqrt[3]{x^2 + 4}$, $a = 2$

15–16 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

15. $f(x) = x + \sqrt{x - 4}$, $[4, \infty)$

16. $g(x) = \frac{x - 1}{3x + 6}$, $(-\infty, -2)$

17–22 Explain why the function is discontinuous at the given number a . Sketch the graph of the function.

17. $f(x) = \frac{1}{x + 2}$ $a = -2$

18. $f(x) = \begin{cases} \frac{1}{x + 2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$ $a = -2$

19. $f(x) = \begin{cases} 1 - x^2 & \text{if } x < 1 \\ 1/x & \text{if } x \geq 1 \end{cases}$ $a = 1$

20. $f(x) = \begin{cases} \frac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$ $a = 1$

21. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$ $a = 0$

22. $f(x) = \begin{cases} \frac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ $a = 3$

23–24 How would you “remove the discontinuity” of f ? In other words, how would you define $f(2)$ in order to make f continuous at 2?

23. $f(x) = \frac{x^2 - x - 2}{x - 2}$ 24. $f(x) = \frac{x^3 - 8}{x^2 - 4}$

25–32 Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

25. $F(x) = \frac{2x^2 - x - 1}{x^2 + 1}$ 26. $G(x) = \frac{x^2 + 1}{2x^2 - x - 1}$

27. $Q(x) = \frac{\sqrt[3]{x - 2}}{x^3 - 2}$ 28. $h(x) = \frac{\sin x}{x + 1}$

29. $h(x) = \cos(1 - x^2)$ 30. $B(x) = \frac{\tan x}{\sqrt{4 - x^2}}$

31. $M(x) = \sqrt{1 + \frac{1}{x}}$ 32. $F(x) = \sin(\cos(\sin x))$

 **33–34** Locate the discontinuities of the function and illustrate by graphing.

33. $y = \frac{1}{1 + \sin x}$ 34. $y = \tan \sqrt{x}$

35–38 Use continuity to evaluate the limit.

35. $\lim_{x \rightarrow 2} x\sqrt{20 - x^2}$ 36. $\lim_{x \rightarrow \pi} \sin(x + \sin x)$

37. $\lim_{x \rightarrow \pi/4} x^2 \tan x$ 38. $\lim_{x \rightarrow 2} x^3 / \sqrt{x^2 + x - 2}$

39–40 Show that f is continuous on $(-\infty, \infty)$.

39. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 1 \\ \sqrt{x - 1} & \text{if } x > 1 \end{cases}$

40. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$

41–43 Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f .

$$41. f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } -1 \leq x < 1 \\ 1/x & \text{if } x \geq 1 \end{cases}$$

$$42. f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 1 \\ 3 - x & \text{if } 1 < x \leq 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

$$43. f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

44. The gravitational force exerted by the planet Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \frac{GM}{R^3} & \text{if } r < R \\ \frac{GM}{r^2} & \text{if } r \geq R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r ?

45. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

46. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$$

47. Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x \rightarrow 2} [3f(x) + f(x)g(x)] = 36$. Find $f(2)$.

48. Let $f(x) = 1/x$ and $g(x) = 1/x^2$.

(a) Find $(f \circ g)(x)$.

(b) Is $f \circ g$ continuous everywhere? Explain.

49. Which of the following functions f has a removable discontinuity at a ? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous at a .

(a) $f(x) = \frac{x^4 - 1}{x - 1}, \quad a = 1$

(b) $f(x) = \frac{x^3 - x^2 - 2x}{x - 2}, \quad a = 2$

(c) $f(x) = \llbracket \sin x \rrbracket, \quad a = \pi$

50. Suppose that a function f is continuous on $[0, 1]$ except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f , one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

51. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

52. Suppose f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

53–56 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

53. $x^4 + x - 3 = 0, \quad (1, 2)$

54. $2/x = x - \sqrt{x}, \quad (2, 3)$

55. $\cos x = x, \quad (0, 1)$

56. $\sin x = x^2 - x, \quad (1, 2)$

57–58 (a) Prove that the equation has at least one real root.

(b) Use your calculator to find an interval of length 0.01 that contains a root.

57. $\cos x = x^3$

58. $x^5 - x^2 + 2x + 3 = 0$

 **59–60** (a) Prove that the equation has at least one real root.

(b) Use your graphing device to find the root correct to three decimal places.

59. $x^5 - x^2 - 4 = 0$

60. $\sqrt{x - 5} = \frac{1}{x + 3}$

61–62 Prove, without graphing, that the graph of the function has at least two x -intercepts in the specified interval.

61. $y = \sin x^3, \quad (1, 2)$

62. $y = x^2 - 3 + 1/x, \quad (0, 2)$

63. Prove that f is continuous at a if and only if

$$\lim_{h \rightarrow 0} f(a + h) = f(a)$$

64. To prove that sine is continuous, we need to show that $\lim_{x \rightarrow a} \sin x = \sin a$ for every real number a . By Exercise 63 an equivalent statement is that

$$\lim_{h \rightarrow 0} \sin(a + h) = \sin a$$

Use (6) to show that this is true.

65. Prove that cosine is a continuous function.

66. (a) Prove Theorem 4, part 3.

(b) Prove Theorem 4, part 5.

67. For what values of x is f continuous?

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

68. For what values of x is g continuous?

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

69. Is there a number that is exactly 1 more than its cube?

70. If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.

71. Show that the function

$$f(x) = \begin{cases} x^4 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $(-\infty, \infty)$.

72. (a) Show that the absolute value function $F(x) = |x|$ is continuous everywhere.
 (b) Prove that if f is a continuous function on an interval, then so is $|f|$.
 (c) Is the converse of the statement in part (b) also true? In other words, if $|f|$ is continuous, does it follow that f is continuous? If so, prove it. If not, find a counterexample.
73. A Tibetan monk leaves the monastery at 7:00 AM and takes his usual path to the top of the mountain, arriving at 7:00 PM. The following morning, he starts at 7:00 AM at the top and takes the same path back, arriving at the monastery at 7:00 PM. Use the Intermediate Value Theorem to show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

1 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

- (a) What is a function? What are its domain and range?

(b) What is the graph of a function?

(c) How can you tell whether a given curve is the graph of a function?
- Discuss four ways of representing a function. Illustrate your discussion with examples.
- (a) What is an even function? How can you tell if a function is even by looking at its graph? Give three examples of an even function.

(b) What is an odd function? How can you tell if a function is odd by looking at its graph? Give three examples of an odd function.
- What is an increasing function?
- What is a mathematical model?
- Give an example of each type of function.

(a) Linear function	(b) Power function
(c) Exponential function	(d) Quadratic function
(e) Polynomial of degree 5	(f) Rational function
- Sketch by hand, on the same axes, the graphs of the following functions.

(a) $f(x) = x$	(b) $g(x) = x^2$
(c) $h(x) = x^3$	(d) $j(x) = x^4$
- Draw, by hand, a rough sketch of the graph of each function.

(a) $y = \sin x$	(b) $y = \cos x$	(c) $y = \tan x$
(d) $y = 1/x$	(e) $y = x $	(f) $y = \sqrt{x}$
- Suppose that f has domain A and g has domain B .
 - What is the domain of $f + g$?
 - What is the domain of fg ?
 - What is the domain of f/g ?
- How is the composite function $f \circ g$ defined? What is its domain?
- Suppose the graph of f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.
 - Shift 2 units upward.
 - Shift 2 units downward.
 - Shift 2 units to the right.
 - Shift 2 units to the left.
 - Reflect about the x -axis.
 - Reflect about the y -axis.
 - Stretch vertically by a factor of 2.
 - Shrink vertically by a factor of 2.
 - Stretch horizontally by a factor of 2.
 - Shrink horizontally by a factor of 2.
- Explain what each of the following means and illustrate with a sketch.

(a) $\lim_{x \rightarrow a} f(x) = L$	(b) $\lim_{x \rightarrow a^+} f(x) = L$	(c) $\lim_{x \rightarrow a^-} f(x) = L$
(d) $\lim_{x \rightarrow a} f(x) = \infty$	(e) $\lim_{x \rightarrow a} f(x) = -\infty$	

13. Describe several ways in which a limit can fail to exist. Illustrate with sketches.
14. What does it mean to say that the line $x = a$ is a vertical asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.
15. State the following Limit Laws.

(a) Sum Law	(b) Difference Law
(c) Constant Multiple Law	(d) Product Law
(e) Quotient Law	(f) Power Law
(g) Root Law	
16. What does the Squeeze Theorem say?
17. (a) What does it mean for f to be continuous at a ?
(b) What does it mean for f to be continuous on the interval $(-\infty, \infty)$? What can you say about the graph of such a function?
18. (a) Give examples of functions that are continuous on $[-1, 1]$.
(b) Give an example of a function that is not continuous on $[0, 1]$.
19. What does the Intermediate Value Theorem say?

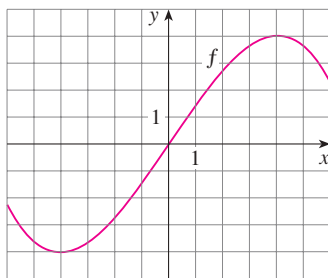
TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

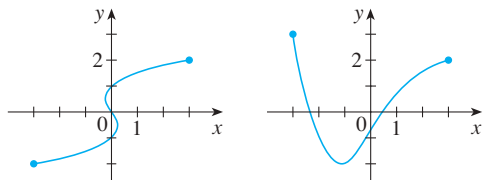
1. If f is a function, then $f(s + t) = f(s) + f(t)$.
2. If $f(s) = f(t)$, then $s = t$.
3. If f is a function, then $f(3x) = 3f(x)$.
4. If $x_1 < x_2$ and f is a decreasing function, then $f(x_1) > f(x_2)$.
5. A vertical line intersects the graph of a function at most once.
6. If x is any real number, then $\sqrt{x^2} = x$.
7. $\lim_{x \rightarrow 4} \left(\frac{2x}{x-4} - \frac{8}{x-4} \right) = \lim_{x \rightarrow 4} \frac{2x}{x-4} - \lim_{x \rightarrow 4} \frac{8}{x-4}$
8. $\lim_{x \rightarrow 1} \frac{x^2 + 6x - 7}{x^2 + 5x - 6} = \frac{\lim_{x \rightarrow 1} (x^2 + 6x - 7)}{\lim_{x \rightarrow 1} (x^2 + 5x - 6)}$
9. $\lim_{x \rightarrow 1} \frac{x-3}{x^2 + 2x - 4} = \frac{\lim_{x \rightarrow 1} (x-3)}{\lim_{x \rightarrow 1} (x^2 + 2x - 4)}$
10. $\frac{x^2 - 9}{x - 3} = x + 3$
11. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} (x + 3)$
12. If $\lim_{x \rightarrow 5} f(x) = 2$ and $\lim_{x \rightarrow 5} g(x) = 0$, then $\lim_{x \rightarrow 5} [f(x)/g(x)]$ does not exist.
13. If $\lim_{x \rightarrow 5} f(x) = 0$ and $\lim_{x \rightarrow 5} g(x) = 0$, then $\lim_{x \rightarrow 5} [f(x)/g(x)]$ does not exist.
14. If neither $\lim_{x \rightarrow a} f(x)$ nor $\lim_{x \rightarrow a} g(x)$ exists, then $\lim_{x \rightarrow a} [f(x) + g(x)]$ does not exist.
15. If $\lim_{x \rightarrow a} f(x)$ exists but $\lim_{x \rightarrow a} g(x)$ does not exist, then $\lim_{x \rightarrow a} [f(x) + g(x)]$ does not exist.
16. If $\lim_{x \rightarrow 6} [f(x)g(x)]$ exists, then the limit must be $f(6)g(6)$.
17. If p is a polynomial, then $\lim_{x \rightarrow b} p(x) = p(b)$.
18. If $\lim_{x \rightarrow 0} f(x) = \infty$ and $\lim_{x \rightarrow 0} g(x) = \infty$, then $\lim_{x \rightarrow 0} [f(x) - g(x)] = 0$.
19. If the line $x = 1$ is a vertical asymptote of $y = f(x)$, then f is not defined at 1.
20. If $f(1) > 0$ and $f(3) < 0$, then there exists a number c between 1 and 3 such that $f(c) = 0$.
21. If f is continuous at 5 and $f(5) = 2$ and $f(4) = 3$, then $\lim_{x \rightarrow 2} f(4x^2 - 11) = 2$.
22. If f is continuous on $[-1, 1]$ and $f(-1) = 4$ and $f(1) = 3$, then there exists a number r such that $|r| < 1$ and $f(r) = \pi$.
23. Let f be a function such that $\lim_{x \rightarrow 0} f(x) = 6$. Then there exists a positive number δ such that if $0 < |x| < \delta$, then $|f(x) - 6| < 1$.
24. If $f(x) > 1$ for all x and $\lim_{x \rightarrow 0} f(x)$ exists, then $\lim_{x \rightarrow 0} f(x) > 1$.
25. The equation $x^{10} - 10x^2 + 5 = 0$ has a root in the interval $(0, 2)$.
26. If f is continuous at a , so is $|f|$.
27. If $|f|$ is continuous at a , so is f .

EXERCISES

1. Let
- f
- be the function whose graph is given.



- (a) Estimate the value of $f(2)$.
 (b) Estimate the values of x such that $f(x) = 3$.
 (c) State the domain of f .
 (d) State the range of f .
 (e) On what interval is f increasing?
 (f) Is f even, odd, or neither even nor odd? Explain.
2. Determine whether each curve is the graph of a function of x . If it is, state the domain and range of the function.



3. If
- $f(x) = x^2 - 2x + 3$
- , evaluate the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

4. Sketch a rough graph of the yield of a crop as a function of the amount of fertilizer used.

5–8 Find the domain and range of the function. Write your answer in interval notation.

5. $f(x) = 2/(3x - 1)$ 6. $g(x) = \sqrt{16 - x^4}$
 7. $y = 1 + \sin x$ 8. $F(t) = 3 + \cos 2t$

9. Suppose that the graph of
- f
- is given. Describe how the graphs of the following functions can be obtained from the graph of
- f
- .

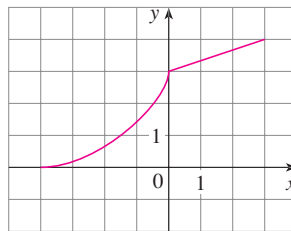
- (a) $y = f(x) + 8$ (b) $y = f(x + 8)$
 (c) $y = 1 + 2f(x)$ (d) $y = f(x - 2) - 2$
 (e) $y = -f(x)$ (f) $y = 3 - f(x)$

10. The graph of
- f
- is given. Draw the graphs of the following functions.

- (a) $y = f(x - 8)$ (b) $y = -f(x)$

(c) $y = 2 - f(x)$

(d) $y = \frac{1}{2}f(x) - 1$



11–16 Use transformations to sketch the graph of the function.

11. $y = (x - 2)^3$

12. $y = 2\sqrt{x}$

13. $y = x^2 - 2x + 2$

14. $y = \frac{1}{x - 1}$

15. $f(x) = -\cos 2x$

16. $f(x) = \begin{cases} 1 + x & \text{if } x < 0 \\ 1 + x^2 & \text{if } x \geq 0 \end{cases}$

17. Determine whether
- f
- is even, odd, or neither even nor odd.

(a) $f(x) = 2x^5 - 3x^2 + 2$

(b) $f(x) = x^3 - x^7$

(c) $f(x) = \cos(x^2)$

(d) $f(x) = 1 + \sin x$

18. Find an expression for the function whose graph consists of the line segment from the point
- $(-2, 2)$
- to the point
- $(-1, 0)$
- together with the top half of the circle with center the origin and radius 1.

19. If
- $f(x) = \sqrt{x}$
- and
- $g(x) = \sin x$
- , find the functions (a)
- $f \circ g$
- , (b)
- $g \circ f$
- , (c)
- $f \circ f$
- , (d)
- $g \circ g$
- , and their domains.

20. Express the function
- $F(x) = 1/\sqrt{x + \sqrt{x}}$
- as a composition of three functions.

21. Life expectancy improved dramatically in the 20th century. The table gives the life expectancy at birth (in years) of males born in the United States. Use a scatter plot to choose an appropriate type of model. Use your model to predict the life span of a male born in the year 2010.

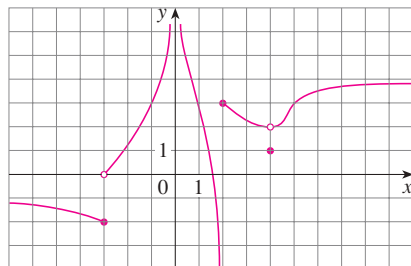
Birth year	Life expectancy	Birth year	Life expectancy
1900	48.3	1960	66.6
1910	51.1	1970	67.1
1920	55.2	1980	70.0
1930	57.4	1990	71.8
1940	62.5	2000	73.0
1950	65.6		

22. A small-appliance manufacturer finds that it costs \$9000 to produce 1000 toaster ovens a week and \$12,000 to produce 1500 toaster ovens a week.

- (a) Express the cost as a function of the number of toaster ovens produced, assuming that it is linear. Then sketch the graph.

- (b) What is the slope of the graph and what does it represent?
 (c) What is the y-intercept of the graph and what does it represent?

23. The graph of f is given.



- (a) Find each limit, or explain why it does not exist.
 (i) $\lim_{x \rightarrow 2^+} f(x)$ (ii) $\lim_{x \rightarrow -3^+} f(x)$ (iii) $\lim_{x \rightarrow -3} f(x)$
 (iv) $\lim_{x \rightarrow 4} f(x)$ (v) $\lim_{x \rightarrow 0} f(x)$ (vi) $\lim_{x \rightarrow 2^-} f(x)$
 (b) State the equations of the vertical asymptotes.
 (c) At what numbers is f discontinuous? Explain.
24. Sketch the graph of an example of a function f that satisfies all of the following conditions:
 $\lim_{x \rightarrow 0^+} f(x) = -2$, $\lim_{x \rightarrow 0^-} f(x) = 1$, $f(0) = -1$,
 $\lim_{x \rightarrow 2^-} f(x) = \infty$, $\lim_{x \rightarrow 2^+} f(x) = -\infty$

25–38 Find the limit.

25. $\lim_{x \rightarrow 0} \cos(x + \sin x)$ 26. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 + 2x - 3}$
 27. $\lim_{x \rightarrow -3} \frac{x^2 - 9}{x^2 + 2x - 3}$ 28. $\lim_{x \rightarrow 1^+} \frac{x^2 - 9}{x^2 + 2x - 3}$
 29. $\lim_{h \rightarrow 0} \frac{(h-1)^3 + 1}{h}$ 30. $\lim_{t \rightarrow 2} \frac{t^2 - 4}{t^3 - 8}$
 31. $\lim_{r \rightarrow 9} \frac{\sqrt{r}}{(r-9)^4}$ 32. $\lim_{v \rightarrow 4^+} \frac{4-v}{|4-v|}$
 33. $\lim_{u \rightarrow 1} \frac{u^4 - 1}{u^3 + 5u^2 - 6u}$ 34. $\lim_{x \rightarrow 3} \frac{\sqrt{x+6} - x}{x^3 - 3x^2}$
 35. $\lim_{s \rightarrow 16} \frac{4 - \sqrt{s}}{s - 16}$ 36. $\lim_{v \rightarrow 2} \frac{v^2 + 2v - 8}{v^4 - 16}$
 37. $\lim_{x \rightarrow 0} \frac{1 - \sqrt{1-x^2}}{x}$
 38. $\lim_{x \rightarrow 1} \left(\frac{1}{x-1} + \frac{1}{x^2 - 3x + 2} \right)$

39. If $2x - 1 \leq f(x) \leq x^2$ for $0 < x < 3$, find $\lim_{x \rightarrow 1} f(x)$.

40. Prove that $\lim_{x \rightarrow 0} x^2 \cos(1/x^2) = 0$.

41–44 Prove the statement using the precise definition of a limit.

41. $\lim_{x \rightarrow 2} (14 - 5x) = 4$

42. $\lim_{x \rightarrow 0} \sqrt[3]{x} = 0$

43. $\lim_{x \rightarrow 2} (x^2 - 3x) = -2$

44. $\lim_{x \rightarrow 4^+} \frac{2}{\sqrt{x-4}} = \infty$

45. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leq x < 3 \\ (x-3)^2 & \text{if } x > 3 \end{cases}$$

(a) Evaluate each limit, if it exists.

(i) $\lim_{x \rightarrow 0^+} f(x)$ (ii) $\lim_{x \rightarrow 0^-} f(x)$ (iii) $\lim_{x \rightarrow 0} f(x)$

(iv) $\lim_{x \rightarrow 3^-} f(x)$ (v) $\lim_{x \rightarrow 3^+} f(x)$ (vi) $\lim_{x \rightarrow 3} f(x)$

(b) Where is f discontinuous?

(c) Sketch the graph of f .

46. Let

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \leq x \leq 2 \\ 2 - x & \text{if } 2 < x \leq 3 \\ x - 4 & \text{if } 3 < x < 4 \\ \pi & \text{if } x \geq 4 \end{cases}$$

(a) For each of the numbers 2, 3, and 4, discover whether g is continuous from the left, continuous from the right, or continuous at the number.

(b) Sketch the graph of g .

47–48 Show that the function is continuous on its domain. State the domain.

47. $h(x) = \sqrt[4]{x} + x^3 \cos x$ 48. $g(x) = \frac{\sqrt{x^2 - 9}}{x^2 - 2}$

49–50 Use the Intermediate Value Theorem to show that there is a root of the equation in the given interval.

49. $x^5 - x^3 + 3x - 5 = 0$, $(1, 2)$

50. $2 \sin x = 3 - 2x$, $(0, 1)$

51. Suppose that $|f(x)| \leq g(x)$ for all x , where $\lim_{x \rightarrow a} g(x) = 0$. Find $\lim_{x \rightarrow a} f(x)$.

52. Let $f(x) = \llbracket x \rrbracket + \llbracket -x \rrbracket$.

(a) For what values of a does $\lim_{x \rightarrow a} f(x)$ exist?

(b) At what numbers is f discontinuous?

Principles of Problem Solving

1 UNDERSTAND THE PROBLEM

There are no hard and fast rules that will ensure success in solving problems. However, it is possible to outline some general steps in the problem-solving process and to give some principles that may be useful in the solution of certain problems. These steps and principles are just common sense made explicit. They have been adapted from George Polya's book *How To Solve It*.

The first step is to read the problem and make sure that you understand it clearly. Ask yourself the following questions:

What is the unknown?

What are the given quantities?

What are the given conditions?

For many problems it is useful to

draw a diagram

and identify the given and required quantities on the diagram.

Usually it is necessary to

introduce suitable notation

In choosing symbols for the unknown quantities we often use letters such as a , b , c , m , n , x , and y , but in some cases it helps to use initials as suggestive symbols; for instance, V for volume or t for time.

2 THINK OF A PLAN

Find a connection between the given information and the unknown that will enable you to calculate the unknown. It often helps to ask yourself explicitly: "How can I relate the given to the unknown?" If you don't see a connection immediately, the following ideas may be helpful in devising a plan.

Try to Recognize Something Familiar Relate the given situation to previous knowledge. Look at the unknown and try to recall a more familiar problem that has a similar unknown.

Try to Recognize Patterns Some problems are solved by recognizing that some kind of pattern is occurring. The pattern could be geometric, or numerical, or algebraic. If you can see regularity or repetition in a problem, you might be able to guess what the continuing pattern is and then prove it.

Use Analogy Try to think of an analogous problem, that is, a similar problem, a related problem, but one that is easier than the original problem. If you can solve the similar, simpler problem, then it might give you the clues you need to solve the original, more difficult problem. For instance, if a problem involves very large numbers, you could first try a similar problem with smaller numbers. Or if the problem involves three-dimensional geometry, you could look for a similar problem in two-dimensional geometry. Or if the problem you start with is a general one, you could first try a special case.

Introduce Something Extra It may sometimes be necessary to introduce something new, an auxiliary aid, to help make the connection between the given and the unknown. For instance, in a problem where a diagram is useful the auxiliary aid could be a new line drawn in a diagram. In a more algebraic problem it could be a new unknown that is related to the original unknown.

Take Cases We may sometimes have to split a problem into several cases and give a different argument for each of the cases. For instance, we often have to use this strategy in dealing with absolute value.

Work Backward Sometimes it is useful to imagine that your problem is solved and work backward, step by step, until you arrive at the given data. Then you may be able to reverse your steps and thereby construct a solution to the original problem. This procedure is commonly used in solving equations. For instance, in solving the equation $3x - 5 = 7$, we suppose that x is a number that satisfies $3x - 5 = 7$ and work backward. We add 5 to each side of the equation and then divide each side by 3 to get $x = 4$. Since each of these steps can be reversed, we have solved the problem.

Establish Subgoals In a complex problem it is often useful to set subgoals (in which the desired situation is only partially fulfilled). If we can first reach these subgoals, then we may be able to build on them to reach our final goal.

Indirect Reasoning Sometimes it is appropriate to attack a problem indirectly. In using proof by contradiction to prove that P implies Q , we assume that P is true and Q is false and try to see why this can't happen. Somehow we have to use this information and arrive at a contradiction to what we absolutely know is true.

Mathematical Induction In proving statements that involve a positive integer n , it is frequently helpful to use the following principle.

Principle of Mathematical Induction Let S_n be a statement about the positive integer n . Suppose that

1. S_1 is true.
2. S_{k+1} is true whenever S_k is true.

Then S_n is true for all positive integers n .

This is reasonable because, since S_1 is true, it follows from condition 2 (with $k = 1$) that S_2 is true. Then, using condition 2 with $k = 2$, we see that S_3 is true. Again using condition 2, this time with $k = 3$, we have that S_4 is true. This procedure can be followed indefinitely.

3 CARRY OUT THE PLAN

In Step 2 a plan was devised. In carrying out that plan we have to check each stage of the plan and write the details that prove that each stage is correct.

4 LOOK BACK

Having completed our solution, it is wise to look back over it, partly to see if we have made errors in the solution and partly to see if we can think of an easier way to solve the problem. Another reason for looking back is that it will familiarize us with the method of solution and this may be useful for solving a future problem. Descartes said, "Every problem that I solved became a rule which served afterwards to solve other problems."

These principles of problem solving are illustrated in the following examples. Before you look at the solutions, try to solve these problems yourself, referring to these Principles of Problem Solving if you get stuck. You may find it useful to refer to this section from time to time as you solve the exercises in the remaining chapters of this book.

As the first example illustrates, it is often necessary to use the problem-solving principle of *taking cases* when dealing with absolute values.

EXAMPLE 1 Solve the inequality $|x - 3| + |x + 2| < 11$.

SOLUTION Recall the definition of absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

It follows that

$$\begin{aligned}|x - 3| &= \begin{cases} x - 3 & \text{if } x - 3 \geq 0 \\ -(x - 3) & \text{if } x - 3 < 0 \end{cases} \\ &= \begin{cases} x - 3 & \text{if } x \geq 3 \\ -x + 3 & \text{if } x < 3 \end{cases}\end{aligned}$$

Similarly

$$\begin{aligned}|x + 2| &= \begin{cases} x + 2 & \text{if } x + 2 \geq 0 \\ -(x + 2) & \text{if } x + 2 < 0 \end{cases} \\ &= \begin{cases} x + 2 & \text{if } x \geq -2 \\ -x - 2 & \text{if } x < -2 \end{cases}\end{aligned}$$

PS Take cases

These expressions show that we must consider three cases:

$$x < -2 \qquad -2 \leq x < 3 \qquad x \geq 3$$

CASE I If $x < -2$, we have

$$\begin{aligned}|x - 3| + |x + 2| &< 11 \\ -x + 3 - x - 2 &< 11 \\ -2x &< 10 \\ x &> -5\end{aligned}$$

CASE II If $-2 \leq x < 3$, the given inequality becomes

$$\begin{aligned}-x + 3 + x + 2 &< 11 \\ 5 &< 11 \quad (\text{always true})\end{aligned}$$

CASE III If $x \geq 3$, the inequality becomes

$$\begin{aligned}x - 3 + x + 2 &< 11 \\ 2x &< 12 \\ x &< 6\end{aligned}$$

Combining cases I, II, and III, we see that the inequality is satisfied when $-5 < x < 6$. So the solution is the interval $(-5, 6)$. 

In the following example we first guess the answer by looking at special cases and recognizing a pattern. Then we prove our conjecture by mathematical induction.

In using the Principle of Mathematical Induction, we follow three steps:

Step 1 Prove that S_n is true when $n = 1$.

Step 2 Assume that S_n is true when $n = k$ and deduce that S_n is true when $n = k + 1$.

Step 3 Conclude that S_n is true for all n by the Principle of Mathematical Induction.

EXAMPLE 2 If $f_0(x) = x/(x + 1)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \dots$, find a formula for $f_n(x)$.

PS Analogy: Try a similar, simpler problem

SOLUTION We start by finding formulas for $f_n(x)$ for the special cases $n = 1, 2$, and 3 .

$$f_1(x) = (f_0 \circ f_0)(x) = f_0(f_0(x)) = f_0\left(\frac{x}{x+1}\right)$$

$$= \frac{\frac{x}{x+1}}{\frac{x}{x+1} + 1} = \frac{\frac{x}{x+1}}{\frac{x+1+x}{x+1}} = \frac{x}{2x+1}$$

$$f_2(x) = (f_0 \circ f_1)(x) = f_0(f_1(x)) = f_0\left(\frac{x}{2x+1}\right)$$

$$= \frac{\frac{x}{2x+1}}{\frac{x}{2x+1} + 1} = \frac{\frac{x}{2x+1}}{\frac{x+2x+1}{2x+1}} = \frac{x}{3x+1}$$

$$f_3(x) = (f_0 \circ f_2)(x) = f_0(f_2(x)) = f_0\left(\frac{x}{3x+1}\right)$$

$$= \frac{\frac{x}{3x+1}}{\frac{x}{3x+1} + 1} = \frac{\frac{x}{3x+1}}{\frac{x+3x+1}{3x+1}} = \frac{x}{4x+1}$$

PS Look for a pattern

We notice a pattern: The coefficient of x in the denominator of $f_n(x)$ is $n + 1$ in the three cases we have computed. So we make the guess that, in general,

$$\boxed{1} \quad f_n(x) = \frac{x}{(n+1)x+1}$$

To prove this, we use the Principle of Mathematical Induction. We have already verified that (1) is true for $n = 1$. Assume that it is true for $n = k$, that is,

$$f_k(x) = \frac{x}{(k+1)x+1}$$

Then
$$f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{x}{(k+1)x+1}\right)$$

$$= \frac{\frac{x}{(k+1)x+1}}{\frac{x}{(k+1)x+1} + 1} = \frac{\frac{x}{(k+1)x+1}}{\frac{x+(k+1)x+1}{(k+1)x+1}} = \frac{x}{(k+2)x+1}$$

This expression shows that (1) is true for $n = k + 1$. Therefore, by mathematical induction, it is true for all positive integers n . ■

In the following example we show how the problem-solving strategy of *introducing something extra* is sometimes useful when we evaluate limits. The idea is to change the variable—to introduce a new variable that is related to the original variable—in such a way as to make the problem simpler. Later, in Section 4.5, we will make more extensive use of this general idea.

EXAMPLE 3 Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1+cx} - 1}{x}$, where c is a constant.

SOLUTION As it stands, this limit looks challenging. In Section 1.6 we evaluated several limits in which both numerator and denominator approached 0. There our strategy was to perform some sort of algebraic manipulation that led to a simplifying cancellation, but here it's not clear what kind of algebra is necessary.

So we introduce a new variable t by the equation

$$t = \sqrt[3]{1+cx}$$

We also need to express x in terms of t , so we solve this equation:

$$t^3 = 1 + cx \quad x = \frac{t^3 - 1}{c} \quad (\text{if } c \neq 0)$$

Notice that $x \rightarrow 0$ is equivalent to $t \rightarrow 1$. This allows us to convert the given limit into one involving the variable t :

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt[3]{1+cx} - 1}{x} &= \lim_{t \rightarrow 1} \frac{t - 1}{(t^3 - 1)/c} \\ &= \lim_{t \rightarrow 1} \frac{c(t - 1)}{t^3 - 1} \end{aligned}$$

The change of variable allowed us to replace a relatively complicated limit by a simpler one of a type that we have seen before. Factoring the denominator as a difference of cubes, we get

$$\begin{aligned} \lim_{t \rightarrow 1} \frac{c(t - 1)}{t^3 - 1} &= \lim_{t \rightarrow 1} \frac{c(t - 1)}{(t - 1)(t^2 + t + 1)} \\ &= \lim_{t \rightarrow 1} \frac{c}{t^2 + t + 1} = \frac{c}{3} \end{aligned}$$

In making the change of variable we had to rule out the case $c = 0$. But if $c = 0$, the function is 0 for all nonzero x and so its limit is 0. Therefore, in all cases, the limit is $c/3$. ■

The following problems are meant to test and challenge your problem-solving skills. Some of them require a considerable amount of time to think through, so don't be discouraged if you can't solve them right away. If you get stuck, you might find it helpful to refer to the discussion of the principles of problem solving.

PROBLEMS

1. Solve the equation $|2x - 1| - |x + 5| = 3$.
2. Solve the inequality $|x - 1| - |x - 3| \geq 5$.
3. Sketch the graph of the function $f(x) = |x^2 - 4|x| + 3|$.

4. Sketch the graph of the function $g(x) = |x^2 - 1| - |x^2 - 4|$.
5. Draw the graph of the equation $x + |x| = y + |y|$.
6. Sketch the region in the plane consisting of all points (x, y) such that
- $$|x - y| + |x| - |y| \leq 2$$
7. The notation $\max\{a, b, \dots\}$ means the largest of the numbers $a, b, \dots$. Sketch the graph of each function.
- $f(x) = \max\{x, 1/x\}$
 - $f(x) = \max\{\sin x, \cos x\}$
 - $f(x) = \max\{x^2, 2 + x, 2 - x\}$
8. Sketch the region in the plane defined by each of the following equations or inequalities.
- $\max\{x, 2y\} = 1$
 - $-1 \leq \max\{x, 2y\} \leq 1$
 - $\max\{x, y^2\} = 1$
9. A driver sets out on a journey. For the first half of the distance she drives at the leisurely pace of 30 mi/h; she drives the second half at 60 mi/h. What is her average speed on this trip?
10. Is it true that $f \circ (g + h) = f \circ g + f \circ h$?
11. Prove that if n is a positive integer, then $7^n - 1$ is divisible by 6.
12. Prove that $1 + 3 + 5 + \dots + (2n - 1) = n^2$.
13. If $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \dots$, find a formula for $f_n(x)$.
14. (a) If $f_0(x) = \frac{1}{2 - x}$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \dots$, find an expression for $f_n(x)$ and use mathematical induction to prove it.
- (b) Graph f_0, f_1, f_2, f_3 on the same screen and describe the effects of repeated composition.



15. Evaluate $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$.

16. Find numbers a and b such that $\lim_{x \rightarrow 0} \frac{\sqrt{ax + b} - 2}{x} = 1$.

17. Evaluate $\lim_{x \rightarrow 0} \frac{|2x - 1| - |2x + 1|}{x}$.

18. The figure shows a point P on the parabola $y = x^2$ and the point Q where the perpendicular bisector of OP intersects the y -axis. As P approaches the origin along the parabola, what happens to Q ? Does it have a limiting position? If so, find it.

19. Evaluate the following limits, if they exist, where $\llbracket x \rrbracket$ denotes the greatest integer function.

(a) $\lim_{x \rightarrow 0} \frac{\llbracket x \rrbracket}{x}$

(b) $\lim_{x \rightarrow 0} x \llbracket 1/x \rrbracket$

20. Sketch the region in the plane defined by each of the following equations.

(a) $\llbracket x \rrbracket^2 + \llbracket y \rrbracket^2 = 1$

(b) $\llbracket x \rrbracket^2 - \llbracket y \rrbracket^2 = 3$

(c) $\llbracket x + y \rrbracket^2 = 1$

(d) $\llbracket x \rrbracket + \llbracket y \rrbracket = 1$

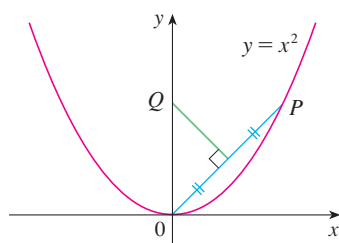


FIGURE FOR PROBLEM 18

21. Find all values of a such that f is continuous on $\mathbb{R}$:

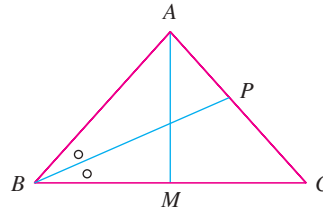
$$f(x) = \begin{cases} x + 1 & \text{if } x \leq a \\ x^2 & \text{if } x > a \end{cases}$$

22. A **fixed point** of a function f is a number c in its domain such that $f(c) = c$. (The function doesn't move c ; it stays fixed.)

- Sketch the graph of a continuous function with domain $[0, 1]$ whose range also lies in $[0, 1]$. Locate a fixed point of f .
- Try to draw the graph of a continuous function with domain $[0, 1]$ and range in $[0, 1]$ that does *not* have a fixed point. What is the obstacle?
- Use the Intermediate Value Theorem to prove that any continuous function with domain $[0, 1]$ and range in $[0, 1]$ must have a fixed point.

23. If $\lim_{x \rightarrow a} [f(x) + g(x)] = 2$ and $\lim_{x \rightarrow a} [f(x) - g(x)] = 1$, find $\lim_{x \rightarrow a} [f(x)g(x)]$.

24. (a) The figure shows an isosceles triangle ABC with $\angle B = \angle C$. The bisector of angle B intersects the side AC at the point P . Suppose that the base BC remains fixed but the altitude $|AM|$ of the triangle approaches 0, so A approaches the midpoint M of BC . What happens to P during this process? Does it have a limiting position? If so, find it.



- Try to sketch the path traced out by P during this process. Then find an equation of this curve and use this equation to sketch the curve.
25. (a) If we start from 0° latitude and proceed in a westerly direction, we can let $T(x)$ denote the temperature at the point x at any given time. Assuming that T is a continuous function of x , show that at any fixed time there are at least two diametrically opposite points on the equator that have exactly the same temperature.
- Does the result in part (a) hold for points lying on any circle on the earth's surface?
 - Does the result in part (a) hold for barometric pressure and for altitude above sea level?